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## Preface to Introduction to Math and Physics

This is a preface to the collection Introduction to Applied Math and Physics

Thanks to OpenStax College, now we have a Physics textbook that is affordable and customizable. This textbook is used in the first year PHY115 Introduction to Math and Physics module in Digipen Institute of Technology (Singapore). The main changes from the College Physics textbook is the number of contents and the arrangement of the contents. PHY 115 deals mainly with classical mechanics, so only the first 10 chapters of College Physics is retained in this textbook. Besides that, few changes in the order of the topics were made to suit the current syllabus in PHY115. It is hoped that with this textbook, students will have more opportunity to own a textbook and get familiar with Physics.

## Preface

Welcome to *College Physics*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

## About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 20 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

## About OpenStax Resources

### Customization

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Because our books are openly licensed, you are free to use the entire book or pick and choose the sections that are most relevant to the needs of your course. Feel free to remix the content by assigning your students certain chapters and sections in your syllabus, in the order that you prefer. You can even provide a direct link in your syllabus to the sections in the web view of your book.

Instructors also have the option of creating a customized version of their OpenStax book. The custom version can be made available to students in low-cost print or digital form through their campus bookstore. Visit your book page on [openstax.org](https://openstax.org) for more information.

## **Errata**

All OpenStax textbooks undergo a rigorous review process. However, like any professional-grade textbook, errors sometimes occur. Since our books are web based, we can make updates periodically when deemed pedagogically necessary. If you have a correction to suggest, submit it through the link on your book page on [openstax.org](https://openstax.org). Subject matter experts review all errata suggestions. OpenStax is committed to remaining transparent about all updates, so you will also find a list of past errata changes on your book page on [openstax.org](https://openstax.org).

## **Format**

You can access this textbook for free in web view or PDF through [openstax.org](https://openstax.org), and in low-cost print and iBooks editions.

## **About *College Physics***

*College Physics* meets standard scope and sequence requirements for a two-semester introductory algebra-based physics course. The text is grounded in real-world examples to help students grasp fundamental physics concepts. It requires knowledge of algebra and some trigonometry, but not calculus. *College Physics* includes learning objectives, concept questions, links to labs and simulations, and ample practice opportunities for traditional physics application problems.

## **Coverage and Scope**

College Physics is organized such that topics are introduced conceptually with a steady progression to precise definitions and analytical applications. The analytical aspect (problem solving) is tied back to the conceptual before moving on to another topic. Each introductory chapter, for example, opens with an engaging photograph relevant to the subject of the chapter and interesting applications that are easy for most students to visualize.

Chapter 1: Introduction: The Nature of Science and Physics

Chapter 2: Kinematics

Chapter 3: Two-Dimensional Kinematics

Chapter 4: Dynamics: Force and Newton's Laws of Motion

Chapter 5: Further Applications of Newton's Laws: Friction, Drag, and Elasticity

Chapter 6: Uniform Circular Motion and Gravitation

Chapter 7: Work, Energy, and Energy Resources

Chapter 8: Linear Momentum and Collisions

Chapter 9: Statics and Torque

Chapter 10: Rotational Motion and Angular Momentum

Chapter 11: Fluid Statics

Chapter 12: Fluid Dynamics and Its Biological and Medical Applications

Chapter 13: Temperature, Kinetic Theory, and the Gas Laws

Chapter 14: Heat and Heat Transfer Methods

Chapter 15: Thermodynamics

Chapter 16: Oscillatory Motion and Waves

Chapter 17: Physics of Hearing

Chapter 18: Electric Charge and Electric Field

Chapter 19: Electric Potential and Electric Field

Chapter 20: Electric Current, Resistance, and Ohm's Law

Chapter 21: Circuits and DC Instruments

Chapter 22: Magnetism

Chapter 23: Electromagnetic Induction, AC Circuits, and Electrical Technologies

Chapter 24: Electromagnetic Waves

Chapter 25: Geometric Optics

Chapter 26: Vision and Optical Instruments

Chapter 27: Wave Optics

Chapter 28: Special Relativity

Chapter 29: Introduction to Quantum Physics  
Chapter 30: Atomic Physics  
Chapter 31: Radioactivity and Nuclear Physics  
Chapter 32: Medical Applications of Nuclear Physics  
Chapter 33: Particle Physics  
Chapter 34: Frontiers of Physics  
Appendix A: Atomic Masses  
Appendix B: Selected Radioactive Isotopes  
Appendix C: Useful Information  
Appendix D: Glossary of Key Symbols and Notation

## **Concepts and Calculations**

The ability to calculate does not guarantee conceptual understanding. In order to unify conceptual, analytical, and calculation skills within the learning process, we have integrated Strategies and Discussions throughout the text.

## **Modern Perspective**

The chapters on modern physics are more complete than many other texts on the market, with an entire chapter devoted to medical applications of nuclear physics and another to particle physics. The final chapter of the text, “Frontiers of Physics,” is devoted to the most exciting endeavors in physics. It ends with a module titled “Some Questions We Know to Ask.”

## **Key Features**

### **Modularity**

This textbook is organized as a collection of modules that can be rearranged and modified to suit the needs of a particular professor or class. That being said, modules often contain references to content in other modules, as most topics in physics cannot be discussed in isolation.

### **Learning Objectives**

Every module begins with a set of learning objectives. These objectives are designed to guide the instructor in deciding what content to include or assign, and to guide the student with respect to what he or she can expect to learn. After completing the module and end-of-module exercises, students should be able to demonstrate mastery of the learning objectives.

### **Call-Outs**

Key definitions, concepts, and equations are called out with a special design treatment. Call-outs are designed to catch readers' attention, to make it clear that a specific term, concept, or equation is particularly important, and to provide easy reference for a student reviewing content.

### **Key Terms**

Key terms are in bold and are followed by a definition in context. Definitions of key terms are also listed in the Glossary, which appears at the end of the module.

### **Worked Examples**

Worked examples have four distinct parts to promote both analytical and conceptual skills. Worked examples are introduced in words, always using some application that should be of interest. This is followed by a Strategy section that emphasizes the concepts involved and how solving the problem

relates to those concepts. This is followed by the mathematical Solution and Discussion.

Many worked examples contain multiple-part problems to help the students learn how to approach normal situations, in which problems tend to have multiple parts. Finally, worked examples employ the techniques of the problem-solving strategies so that students can see how those strategies succeed in practice as well as in theory.

### **Problem-Solving Strategies**

Problem-solving strategies are first presented in a special section and subsequently appear at crucial points in the text where students can benefit most from them. Problem-solving strategies have a logical structure that is reinforced in the worked examples and supported in certain places by line drawings that illustrate various steps.

### **Misconception Alerts**

Students come to physics with preconceptions from everyday experiences and from previous courses. Some of these preconceptions are misconceptions, and many are very common among students and the general public. Some are inadvertently picked up through misunderstandings of lectures and texts. The Misconception Alerts feature is designed to point these out and correct them explicitly.

### **Take-Home Investigations**

Take Home Investigations provide the opportunity for students to apply or explore what they have learned with a hands-on activity.

### **Things Great and Small**



In these special topic essays, macroscopic phenomena (such as air pressure) are explained with submicroscopic phenomena (such as atoms bouncing off walls). These essays support the modern perspective by describing aspects of modern physics before they are formally treated in later chapters. Connections are also made between apparently disparate phenomena.

### **Simulations**

Where applicable, students are directed to the interactive PHeT physics simulations developed by the University of Colorado. There they can further explore the physics concepts they have learned about in the module.

### **Summary**

Module summaries are thorough and functional and present all important definitions and equations. Students are able to find the definitions of all terms and symbols as well as their physical relationships. The structure of the summary makes plain the fundamental principles of the module or collection and serves as a useful study guide.

### **Glossary**

At the end of every module or chapter is a Glossary containing definitions of all of the key terms in the module or chapter.

### **End-of-Module Problems**

At the end of every chapter is a set of Conceptual Questions and/or skills-based Problems & Exercises. Conceptual Questions challenge students' ability to explain what they have learned conceptually, independent of the mathematical details. Problems & Exercises challenge students to apply both concepts and skills to solve mathematical physics problems. Online,

every other problem includes an answer that students can reveal immediately by clicking on a “Show Solution” button.

In addition to traditional skills-based problems, there are three special types of end-of-module problems: Integrated Concept Problems, Unreasonable Results Problems, and Construct Your Own Problems. All of these problems are indicated with a subtitle preceding the problem.

### **Integrated Concept Problems**

In Integrated Concept Problems, students are asked to apply what they have learned about two or more concepts to arrive at a solution to a problem. These problems require a higher level of thinking because, before solving a problem, students have to recognize the combination of strategies required to solve it.

### **Unreasonable Results**

In Unreasonable Results Problems, students are challenged to not only apply concepts and skills to solve a problem, but also to analyze the answer with respect to how likely or realistic it really is. These problems contain a premise that produces an unreasonable answer and are designed to further emphasize that properly applied physics must describe nature accurately and is not simply the process of solving equations.

### **Construct Your Own Problem**

These problems require students to construct the details of a problem, justify their starting assumptions, show specific steps in the problem’s solution, and finally discuss the meaning of the result. These types of problems relate well to both conceptual and analytical aspects of physics, emphasizing that physics must describe nature. Often they involve an integration of topics from more than one chapter. Unlike other problems, solutions are not provided since there is no single correct answer.

Instructors should feel free to direct students regarding the level and scope of their considerations. Whether the problem is solved and described correctly will depend on initial assumptions.

## **Additional Resources**

### **Student and Instructor Resources**

We've compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your [openstax.org](https://openstax.org) log-in. Take advantage of these resources to supplement your OpenStax book.

### **Partner Resources**

OpenStax Partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on [openstax.org](https://openstax.org).

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# Introduction to Science and the Realm of Physics, Physical Quantities, and Units

class="introduction"

Galaxies are  
as immense  
as atoms are  
small. Yet the  
same laws of  
physics  
describe  
both, and all  
the rest of  
nature—an  
indication of  
the  
underlying  
unity in the  
universe. The  
laws of  
physics are  
surprisingly  
few in  
number,  
implying an  
underlying  
simplicity to  
nature's  
apparent  
complexity.  
(credit:  
NASA, JPL-  
Caltech, P.  
Barmby,  
Harvard-  
Smithsonian  
Center for

## Astrophysics )



What is your first reaction when you hear the word “physics”? Did you imagine working through difficult equations or memorizing formulas that seem to have no real use in life outside the physics classroom? Many people come to the subject of physics with a bit of fear. But as you begin your exploration of this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

For example, take a look at the image above. This image is of the Andromeda Galaxy, which contains billions of individual stars, huge clouds of gas, and dust. Two smaller galaxies are also visible as bright blue spots in the background. At a staggering 2.5 million light years from the Earth, this galaxy is the nearest one to our own galaxy (which is called the Milky Way). The stars and planets that make up Andromeda might seem to be the furthest thing from most people’s regular, everyday lives. But Andromeda is a great starting point to think about the forces that hold together the universe. The forces that cause Andromeda to act as it does are the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply raise the walls for a new home. The same gravity that causes the stars of Andromeda to rotate and revolve also causes water to flow over hydroelectric dams here on Earth. Tonight, take a moment to look up at the stars. The forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater

understanding of the interconnectedness of everything we can see and know in this universe.

Think now about all of the technological devices that you use on a regular basis. Computers, smart phones, GPS systems, MP3 players, and satellite radio might come to mind. Next, think about the most exciting modern technologies that you have heard about in the news, such as trains that levitate above tracks, “invisibility cloaks” that bend light around them, and microscopic robots that fight cancer cells in our bodies. All of these groundbreaking advancements, commonplace or unbelievable, rely on the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path and a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, physics principles are propelling new, exciting technologies, and these principles are applied in a wide range of careers.

In this text, you will begin to explore the history of the formal study of physics, beginning with natural philosophy and the ancient Greeks, and leading up through a review of Sir Isaac Newton and the laws of physics that bear his name. You will also be introduced to the standards scientists use when they study physical quantities and the interrelated system of measurements most of the scientific community uses to communicate in a single mathematical language. Finally, you will study the limits of our ability to be accurate and precise, and the reasons scientists go to painstaking lengths to be as clear as possible regarding their own limitations.

## Physics: An Introduction

- Explain the difference between a principle and a law.
- Explain the difference between a model and a theory.



The flight formations of migratory birds such as Canada geese are governed by the laws of physics.  
(credit: David Merrett)

The physical universe is enormously complex in its detail. Every day, each of us observes a great variety of objects and phenomena. Over the centuries, the curiosity of the human race has led us collectively to explore and catalog a tremendous wealth of information. From the flight of birds to the colors of flowers, from lightning to gravity, from quarks to clusters of galaxies, from the flow of time to the mystery of the creation of the universe, we have asked questions and assembled huge arrays of facts. In the face of all these details, we have discovered that a surprisingly small and unified set of physical laws can explain what we observe. As humans, we make generalizations and seek order. We have found that nature is remarkably cooperative—it exhibits the *underlying order and simplicity* we so value.

It is the underlying order of nature that makes science in general, and physics in particular, so enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be



converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

The unifying aspect of physical laws and the basic simplicity of nature form the underlying themes of this text. In learning to apply these laws, you will, of course, study the most important topics in physics. More importantly, you will gain analytical abilities that will enable you to apply these laws far beyond the scope of what can be included in a single book. These analytical skills will help you to excel academically, and they will also help you to think critically in any professional career you choose to pursue. This module discusses the realm of physics (to define what physics is), some applications of physics (to illustrate its relevance to other disciplines), and more precisely what constitutes a physical law (to illuminate the importance of experimentation to theory).

## Science and the Realm of Physics

Science consists of the theories and laws that are the general truths of nature as well as the body of knowledge they encompass. Scientists are continually trying to expand this body of knowledge and to perfect the expression of the laws that describe it. **Physics** is concerned with describing the interactions of energy, matter, space, and time, and it is especially interested in what fundamental mechanisms underlie every phenomenon. The concern for describing the basic phenomena in nature essentially defines the *realm of physics*.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics. Consider a smart phone ([\[link\]](#)). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and

circuit layout when building the smart phone. Next, consider a GPS system. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another.



The Apple  
“iPhone” is a  
common  
smart phone  
with a GPS  
function.

Physics  
describes the  
way that  
electricity  
flows through  
the circuits of  
this device.  
Engineers use  
their  
knowledge of  
physics to  
construct an

iPhone with features that consumers will enjoy. One specific feature of an iPhone is the GPS function. GPS uses physics equations to determine the driving time between two locations on a map. (credit: @gletham GIS, Social, Mobile Tech Images)

## **Applications of Physics**

You need not be a scientist to use physics. On the contrary, knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. (See [\[link\]](#) and [\[link\]](#).) Physics allows you to understand the hazards of radiation and rationally evaluate these hazards more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car's ignition system as well as the transmission of electrical signals through our body's nervous system are

much easier to understand when you think about them in terms of basic physics.

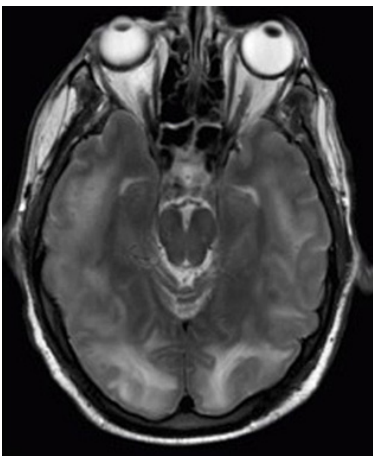
Physics is the foundation of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—is rooted in atomic and molecular physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability, and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes ([\[link\]](#) and [\[link\]](#)). On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as x-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics; for example, cancer radiotherapy uses ionizing radiation. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

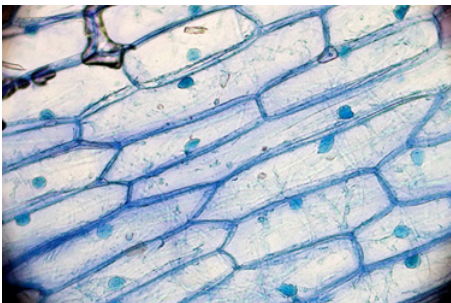
It is not necessary to formally study all applications of physics. What is most useful is knowledge of the basic laws of physics and a skill in the analytical methods for applying them. The study of physics also can improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences, and the study of physics makes other sciences easier to understand.



The laws of physics help us understand how common appliances work. For example, the laws of physics can help explain how microwave ovens heat up food, and they also help us understand why it is dangerous to place metal objects in a microwave oven. (credit: MoneyBlogNewz)

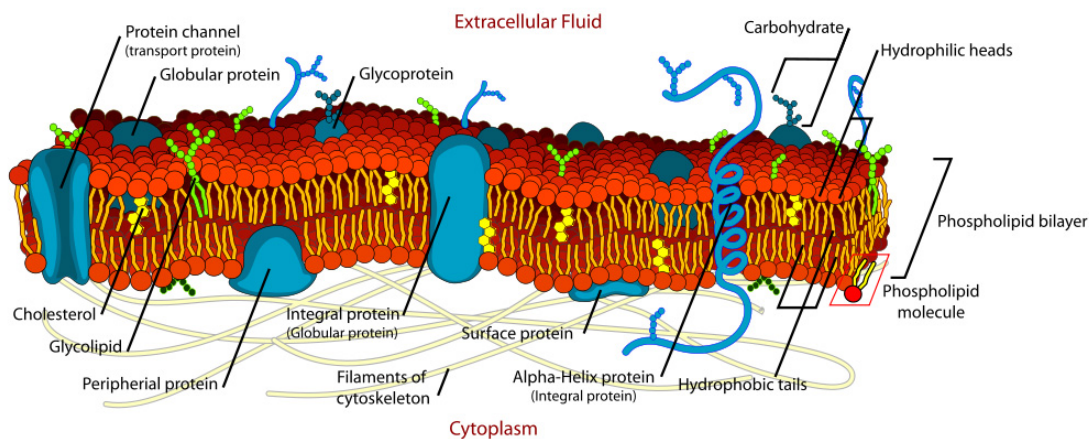


These two applications of physics have more in common than meets the eye. Microwave ovens use electromagnetic waves to heat food. Magnetic resonance imaging (MRI) also uses electromagnetic waves to yield an image of the brain, from which the exact location of tumors can be determined.  
(credit: Rashmi Chawla, Daniel Smith, and Paul E. Marik)



Physics, chemistry,

and biology help describe the properties of cell walls in plant cells, such as the onion cells seen here. (credit: Umberto Salvagnin)

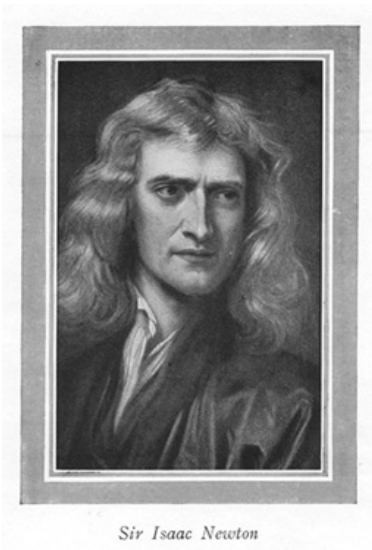


An artist's rendition of the structure of a cell membrane. Membranes form the boundaries of animal cells and are complex in structure and function. Many of the most fundamental properties of life, such as the firing of nerve cells, are related to membranes. The disciplines of biology, chemistry, and physics all help us understand the membranes of animal cells. (credit: Mariana Ruiz)

## Models, Theories, and Laws; The Role of Experimentation

The laws of nature are concise descriptions of the universe around us; they are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not

create them and so cannot change them. We can only discover and understand them. Their discovery is a very human endeavor, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort. (See [\[link\]](#) and [\[link\]](#).) The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be.



**Isaac Newton**  
(1642–1727) was very reluctant to publish his revolutionary work and had to be convinced to do so. In his later years, he stepped down from his academic post and became exchequer of the Royal Mint. He took this post



seriously,  
inventing reeding  
(or creating  
ridges) on the  
edge of coins to  
prevent  
unscrupulous  
people from  
trimming the  
silver off of them  
before using them  
as currency.  
(credit: Arthur  
Shuster and  
Arthur E. Shipley:  
*Britain's Heritage  
of Science*.  
London, 1917.)



**Marie Curie**  
(1867–1934)  
sacrificed

monetary assets  
to help finance  
her early  
research and  
damaged her  
physical well-  
being with  
radiation  
exposure. She is  
the only person  
to win Nobel  
prizes in both  
physics and  
chemistry. One  
of her daughters  
also won a  
Nobel Prize.  
(credit:  
Wikimedia  
Commons)

We all are curious to some extent. We look around, make generalizations, and try to understand what we see—for example, we look up and wonder whether one type of cloud signals an oncoming storm. As we become serious about exploring nature, we become more organized and formal in collecting and analyzing data. We attempt greater precision, perform controlled experiments (if we can), and write down ideas about how the data may be organized and unified. We then formulate models, theories, and laws based on the data we have collected and analyzed to generalize and communicate the results of these experiments.

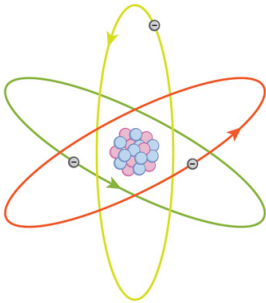
A **model** is a representation of something that is often too difficult (or impossible) to display directly. While a model is justified with experimental proof, it is only accurate under limited situations. An example is the planetary model of the atom in which electrons are pictured as orbiting the

nucleus, analogous to the way planets orbit the Sun. (See [\[link\]](#).) We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases (atomic spectra). Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation. A **theory** is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena, whereas others do not. Newton's theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behavior of gases.

A **law** uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation *law* is reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force, mass, and acceleration by the simple equation  $\mathbf{F} = m\mathbf{a}$ . A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Less broadly applicable statements are usually called principles (such as Pascal's principle, which is applicable only in fluids), but the distinction

between laws and principles often is not carefully made.



What is a  
model?

This  
planetary  
model of  
the atom  
shows  
electrons  
orbiting the  
nucleus. It  
is a  
drawing  
that we use  
to form a  
mental  
image of  
the atom  
that we  
cannot see  
directly  
with our  
eyes  
because it  
is too  
small.

**Note:****Models, Theories, and Laws**

Models, theories, and laws are used to help scientists analyze the data they have already collected. However, often after a model, theory, or law has been developed, it points scientists toward new discoveries they would not otherwise have made.

The models, theories, and laws we devise sometimes *imply the existence of objects or phenomena as yet unobserved*. These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if *experiment* does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment in order to confirm a law in every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law, then the law must be modified or overthrown completely.

The study of science in general and physics in particular is an adventure much like the exploration of uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

**Note:****The Scientific Method**

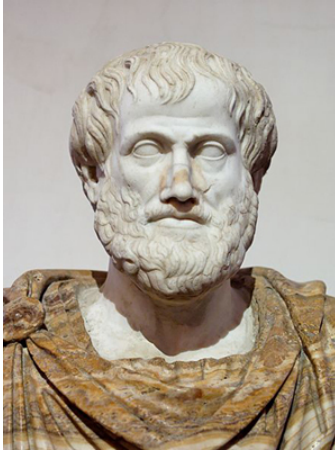
As scientists inquire and gather information about the world, they follow a process called the **scientific method**. This process typically begins with an observation and question that the scientist will research. Next, the scientist

typically performs some research about the topic and then devises a hypothesis. Then, the scientist will test the hypothesis by performing an experiment. Finally, the scientist analyzes the results of the experiment and draws a conclusion. Note that the scientific method can be applied to many situations that are not limited to science, and this method can be modified to suit the situation.

Consider an example. Let us say that you try to turn on your car, but it will not start. You undoubtedly wonder: Why will the car not start? You can follow a scientific method to answer this question. First off, you may perform some research to determine a variety of reasons why the car will not start. Next, you will state a hypothesis. For example, you may believe that the car is not starting because it has no engine oil. To test this, you open the hood of the car and examine the oil level. You observe that the oil is at an acceptable level, and you thus conclude that the oil level is not contributing to your car issue. To troubleshoot the issue further, you may devise a new hypothesis to test and then repeat the process again.

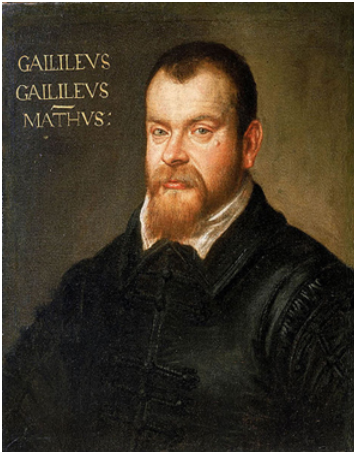
## The Evolution of Natural Philosophy into Modern Physics

Physics was not always a separate and distinct discipline. It remains connected to other sciences to this day. The word *physics* comes from Greek, meaning nature. The study of nature came to be called “natural philosophy.” From ancient times through the Renaissance, natural philosophy encompassed many fields, including astronomy, biology, chemistry, physics, mathematics, and medicine. Over the last few centuries, the growth of knowledge has resulted in ever-increasing specialization and branching of natural philosophy into separate fields, with physics retaining the most basic facets. (See [\[link\]](#), [\[link\]](#), and [\[link\]](#).) Physics as it developed from the Renaissance to the end of the 19th century is called **classical physics**. It was transformed into modern physics by revolutionary discoveries made starting at the beginning of the 20th century.



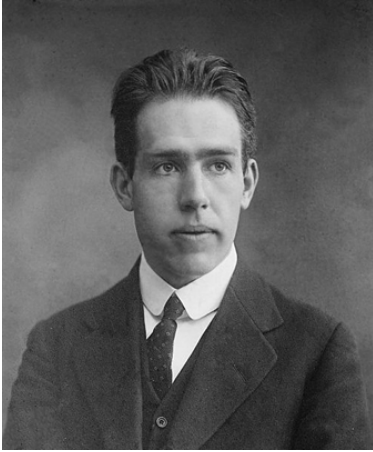
Over the centuries, natural philosophy has evolved into more specialized disciplines, as illustrated by the contributions of some of the greatest minds in history. The Greek philosopher **Aristotle** (384–322 B.C.) wrote on a broad range of topics including physics, animals, the soul, politics, and poetry.  
(credit: Jastrow

(2006)/Ludovisi  
Collection)



**Galileo Galilei**  
(1564–1642) laid  
the foundation of  
modern  
experimentation  
and made  
contributions in  
mathematics,  
physics, and  
astronomy.  
(credit:  
Domenico  
Tintoretto)





**Niels Bohr**  
(1885–1962)  
made  
fundamental  
contributions to  
the development  
of quantum  
mechanics, one  
part of modern  
physics. (credit:  
United States  
Library of  
Congress Prints  
and Photographs  
Division)

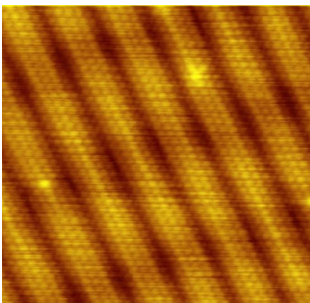
Classical physics is not an exact description of the universe, but it is an excellent approximation under the following conditions: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields, such as the field generated by the Earth, can be involved. Because humans live under such circumstances, classical physics seems intuitively reasonable, while many aspects of modern physics seem bizarre. This is why models are so useful in modern physics—they let us

conceptualize phenomena we do not ordinarily experience. We can relate to models in human terms and visualize what happens when objects move at high speeds or imagine what objects too small to observe with our senses might be like. For example, we can understand an atom's properties because we can picture it in our minds, although we have never seen an atom with our eyes. New tools, of course, allow us to better picture phenomena we cannot see. In fact, new instrumentation has allowed us in recent years to actually “picture” the atom.

**Note:**

**Limits on the Laws of Classical Physics**

For the laws of classical physics to apply, the following criteria must be met: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields (such as the field generated by the Earth) can be involved.



Using a  
scanning  
tunneling  
microscope  
(STM),  
scientists can  
see the  
individual  
atoms that

compose this  
sheet of gold.  
(credit:  
Erwinrossen)

Some of the most spectacular advances in science have been made in modern physics. Many of the laws of classical physics have been modified or rejected, and revolutionary changes in technology, society, and our view of the universe have resulted. Like science fiction, modern physics is filled with fascinating objects beyond our normal experiences, but it has the advantage over science fiction of being very real. Why, then, is the majority of this text devoted to topics of classical physics? There are two main reasons: Classical physics gives an extremely accurate description of the universe under a wide range of everyday circumstances, and knowledge of classical physics is necessary to understand modern physics.

**Modern physics** itself consists of the two revolutionary theories, relativity and quantum mechanics. These theories deal with the very fast and the very small, respectively. **Relativity** must be used whenever an object is traveling at greater than about 1% of the speed of light or experiences a strong gravitational field such as that near the Sun. **Quantum mechanics** must be used for objects smaller than can be seen with a microscope. The combination of these two theories is *relativistic quantum mechanics*, and it describes the behavior of small objects traveling at high speeds or experiencing a strong gravitational field. Relativistic quantum mechanics is the best universally applicable theory we have. Because of its mathematical complexity, it is used only when necessary, and the other theories are used whenever they will produce sufficiently accurate results. We will find, however, that we can do a great deal of modern physics with the algebra and trigonometry used in this text.

**Exercise:**

**Check Your Understanding**

**Problem:**

A friend tells you he has learned about a new law of nature. What can you know about the information even before your friend describes the law? How would the information be different if your friend told you he had learned about a scientific theory rather than a law?

---

**Solution:**

Without knowing the details of the law, you can still infer that the information your friend has learned conforms to the requirements of all laws of nature: it will be a concise description of the universe around us; a statement of the underlying rules that all natural processes follow. If the information had been a theory, you would be able to infer that the information will be a large-scale, broadly applicable generalization.

**Note:**

PhET Explorations: Equation Grapher

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g.  $y = bx$ ) to see how they add to generate the polynomial curve.

[https://phet.colorado.edu/sims/equation-grapher/equation-grapher\\_en.html](https://phet.colorado.edu/sims/equation-grapher/equation-grapher_en.html)

**Summary**

- Science seeks to discover and describe the underlying order and simplicity in nature.
- Physics is the most basic of the sciences, concerning itself with energy, matter, space and time, and their interactions.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.

## Conceptual Questions

### Exercise:

#### Problem:

Models are particularly useful in relativity and quantum mechanics, where conditions are outside those normally encountered by humans. What is a model?

### Exercise:

**Problem:** How does a model differ from a theory?

### Exercise:

#### Problem:

If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?

### Exercise:

**Problem:** What determines the validity of a theory?

### Exercise:

#### Problem:

Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?

### Exercise:

#### Problem:

Can the validity of a model be limited, or must it be universally valid? How does this compare to the required validity of a theory or a law?

### Exercise:

**Problem:**

Classical physics is a good approximation to modern physics under certain circumstances. What are they?

**Exercise:**

**Problem:** When is it *necessary* to use relativistic quantum mechanics?

**Exercise:****Problem:**

Can classical physics be used to accurately describe a satellite moving at a speed of 7500 m/s? Explain why or why not.

**Glossary**

classical physics

physics that was developed from the Renaissance to the end of the 19th century

physics

the science concerned with describing the interactions of energy, matter, space, and time; it is especially interested in what fundamental mechanisms underlie every phenomenon

model

representation of something that is often too difficult (or impossible) to display directly

theory

an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers

law

a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence

and repeated experiments

scientific method

a method that typically begins with an observation and question that the scientist will research; next, the scientist typically performs some research about the topic and then devises a hypothesis; then, the scientist will test the hypothesis by performing an experiment; finally, the scientist analyzes the results of the experiment and draws a conclusion

modern physics

the study of relativity, quantum mechanics, or both

relativity

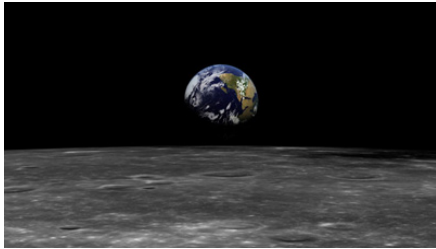
the study of objects moving at speeds greater than about 1% of the speed of light, or of objects being affected by a strong gravitational field

quantum mechanics

the study of objects smaller than can be seen with a microscope

## Physical Quantities and Units

- Perform unit conversions both in the SI and English units.
- Explain the most common prefixes in the SI units and be able to write them in scientific notation.

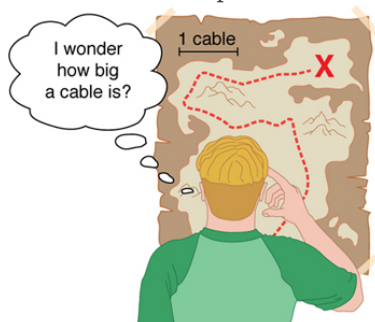


The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distances from Earth to other celestial bodies. (credit: NASA)

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—all physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

We define a **physical quantity** either by *specifying how it is measured* or by *stating how it is calculated* from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define *average speed* by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See [\[link\]](#).)





Distances given in  
unknown units are  
maddeningly useless.

There are two major systems of units used in the world: **SI units** (also known as the metric system) and **English units** (also known as the customary or imperial system). **English units** were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym “SI” is derived from the French *Système International*.

## SI Units: Fundamental and Derived Units

[\[link\]](#) gives the fundamental SI units that are used throughout this textbook. This text uses non-SI units in a few applications where they are in very common use, such as the measurement of blood pressure in millimeters of mercury (mm Hg). Whenever non-SI units are discussed, they will be tied to SI units through conversions.

Length	Mass	Time	Electric Current
meter (m)	kilogram (kg)	second (s)	ampere (A)

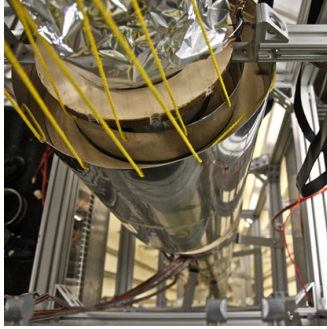
### Fundamental SI Units

It is an intriguing fact that some physical quantities are more fundamental than others and that the most fundamental physical quantities can be defined *only* in terms of the procedure used to measure them. The units in which they are measured are thus called **fundamental units**. In this textbook, the fundamental physical quantities are taken to be length, mass, time, and electric current. (Note that electric current will not be introduced until much later in this text.) All other physical quantities, such as force and electric charge, can be expressed as algebraic combinations of length, mass, time, and current (for example, speed is length divided by time); these units are called **derived units**.

## Units of Time, Length, and Mass: The Second, Meter, and Kilogram

### The Second

The SI unit for time, the **second**(abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a non-varying, or constant, physical phenomenon (because the solar day is getting longer due to very gradual slowing of the Earth’s rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967 the second was redefined as the time required for 9,192,631,770 of these vibrations. (See [\[link\]](#).) Accuracy in the fundamental units is essential, because all measurements are ultimately expressed in terms of fundamental units and can be no more accurate than are the fundamental units themselves.



An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image is looking down from the top of an atomic fountain nearly 30 feet tall!  
(credit: Steve Jurvetson/Flickr)

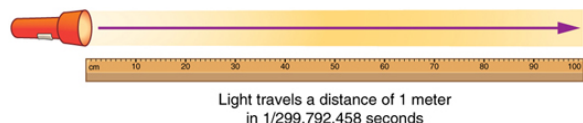
## The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more accurate and precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum-iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its present definition (partly for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second. (See [\[link\]](#).) This change defines the speed of light to be exactly 299,792,458 meters per second. The length of the meter will change if the speed of light is someday measured with greater accuracy.

## The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); it is defined to be the mass of a platinum-iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. Exact replicas of the standard kilogram are also kept at the United States' National Institute of Standards

and Technology, or NIST, located in Gaithersburg, Maryland outside of Washington D.C., and at other locations around the world. The determination of all other masses can be ultimately traced to a comparison with the standard mass.



The meter is defined to be the distance light travels in  $1/299,792,458$  of a second in a vacuum. Distance traveled is speed multiplied by time.

Electric current and its accompanying unit, the ampere, will be introduced in [Introduction to Electric Current, Resistance, and Ohm's Law](#) when electricity and magnetism are covered. The initial modules in this textbook are concerned with mechanics, fluids, heat, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the fundamental units of length, mass, and time.

## Metric Prefixes

SI units are part of the **metric system**. The metric system is convenient for scientific and engineering calculations because the units are categorized by factors of 10. [\[link\]](#) gives metric prefixes and symbols used to denote various factors of 10.

Metric systems have the advantage that conversions of units involve only powers of 10. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on. Another advantage of the metric system is that the same unit can be used over extremely large ranges of values simply by using an appropriate metric prefix. For example, distances in meters are suitable in construction, while distances in kilometers are appropriate for air travel, and the tiny measure of nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications.

The term **order of magnitude** refers to the scale of a value expressed in the metric system. Each power of 10 in the metric system represents a different order of magnitude. For example,  $10^1$ ,  $10^2$ ,  $10^3$ , and so forth are all different orders of magnitude. All quantities that can be expressed as a product of a specific power of 10 are said to be of the *same* order of magnitude. For example, the number 800 can be written as  $8 \times 10^2$ , and the number 450 can be written as  $4.5 \times 10^2$ . Thus, the numbers 800 and 450 are of the same order of magnitude:  $10^2$ . Order of magnitude can be thought of as a ballpark estimate for the scale of a value. The diameter of an atom is on the order of  $10^{-9}$  m, while the diameter of the Sun is on the order of  $10^9$  m.

### Note:

The Quest for Microscopic Standards for Basic Units

The fundamental units described in this chapter are those that produce the greatest accuracy and precision in measurement. There is a sense among physicists that, because there is an underlying microscopic substructure to matter, it would be most satisfying to base our standards of measurement on microscopic objects and fundamental physical phenomena such as the speed of light. A microscopic standard has been accomplished for the standard of time, which is based on the oscillations of the cesium atom.

The standard for length was once based on the wavelength of light (a small-scale length) emitted by a certain type of atom, but it has been supplanted by the more precise measurement of the speed of light. If it becomes possible to measure the mass of atoms or a particular arrangement of atoms such as a silicon sphere to greater precision than the kilogram standard, it may become possible to base mass measurements on the small scale. There are also possibilities that electrical phenomena on the small scale may someday allow us to base a unit of charge on the charge of electrons and protons, but at present current and charge are related to large-scale currents and forces between wires.

Prefix	Symbol	Value <sup><a href="#">[footnote]</a></sup> See <a href="#">Appendix A</a> for a discussion of powers of 10.	Example (some are approximate)			
exa	E	$10^{18}$	exameter	Em	$10^{18}$ m	distance light travels in a century
peta	P	$10^{15}$	petasecond	Ps	$10^{15}$ s	30 million years
tera	T	$10^{12}$	terawatt	TW	$10^{12}$ W	powerful laser output
giga	G	$10^9$	gigahertz	GHz	$10^9$ Hz	a microwave frequency
mega	M	$10^6$	megacurie	MCi	$10^6$ Ci	high radioactivity
kilo	k	$10^3$	kilometer	km	$10^3$ m	about 6/10 mile
hecto	h	$10^2$	hectoliter	hL	$10^2$ L	26 gallons

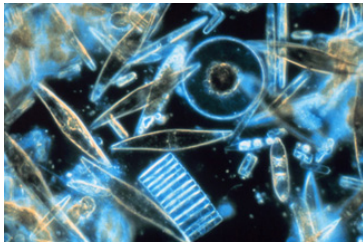
Prefix	Symbol	Value <sup><a href="#">[footnote]</a></sup> See <a href="#">Appendix A</a> for a discussion of powers of 10.	Example (some are approximate)			
deka	da	$10^1$	dekagram	dag	$10^1$ g	teaspoon of butter
—	—	$10^0$ (=1)				
deci	d	$10^{-1}$	deciliter	dL	$10^{-1}$ L	less than half a soda
centi	c	$10^{-2}$	centimeter	cm	$10^{-2}$ m	fingertip thickness
milli	m	$10^{-3}$	millimeter	mm	$10^{-3}$ m	flea at its shoulders
micro	$\mu$	$10^{-6}$	micrometer	$\mu\text{m}$	$10^{-6}$ m	detail in microscope
nano	n	$10^{-9}$	nanogram	ng	$10^{-9}$ g	small speck of dust
pico	p	$10^{-12}$	picofarad	pF	$10^{-12}$ F	small capacitor in radio
femto	f	$10^{-15}$	femtometer	fm	$10^{-15}$ m	size of a proton
atto	a	$10^{-18}$	attosecond	as	$10^{-18}$ s	time light crosses an atom

Metric Prefixes for Powers of 10 and their Symbols

### Known Ranges of Length, Mass, and Time

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times in [\[link\]](#). Examination of this table will give you some

feeling for the range of possible topics and numerical values. (See [\[link\]](#) and [\[link\]](#).)

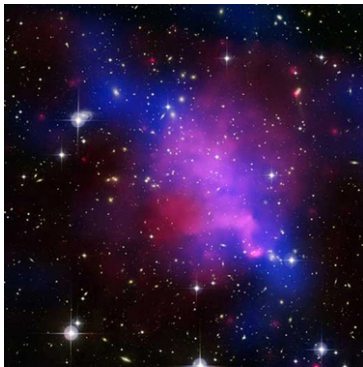


Tiny phytoplankton  
swims among crystals of  
ice in the Antarctic Sea.

They range from a few  
micrometers to as much  
as 2 millimeters in length.

(credit: Prof. Gordon T.

Taylor, Stony Brook  
University; NOAA Corps  
Collections)



Galaxies collide 2.4  
billion light years away  
from Earth. The

tremendous range of  
observable phenomena in  
nature challenges the  
imagination. (credit:

NASA/CXC/UVic./A.

Mahdavi et al.

Optical/lensing:  
CFHT/UVic./H. Hoekstra  
et al.)

Unit Conversion and Dimensional Analysis

It is often necessary to convert from one type of unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles.

Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km).

The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in *meters* and we want to convert to *kilometers*.

Next, we need to determine a **conversion factor** relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

Equation:

80 m × (1 km / 1000 m) = 0.080 km.

Note that the unwanted m unit cancels, leaving only the desired km unit. You can use this method to convert between any types of unit.

Click [\[link\]](#) for a more complete list of conversion factors.

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
10 <sup>-18</sup>	Present experimental limit to smallest observable detail	10 <sup>-30</sup>	Mass of an electron (9.11 × 10 <sup>-31</sup> kg)	10 <sup>-23</sup>	Time for light to cross a proton
10 <sup>-15</sup>	Diameter of a proton	10 <sup>-27</sup>	Mass of a hydrogen atom (1.67 × 10 <sup>-27</sup> kg)	10 <sup>-22</sup>	Mean life of an extremely unstable nucleus

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
$10^{-14}$	Diameter of a uranium nucleus	$10^{-15}$	Mass of a bacterium	$10^{-15}$	Time for one oscillation of visible light
$10^{-10}$	Diameter of a hydrogen atom	$10^{-5}$	Mass of a mosquito	$10^{-13}$	Time for one vibration of an atom in a solid
$10^{-8}$	Thickness of membranes in cells of living organisms	$10^{-2}$	Mass of a hummingbird	$10^{-8}$	Time for one oscillation of an FM radio wave
$10^{-6}$	Wavelength of visible light	1	Mass of a liter of water (about a quart)	$10^{-3}$	Duration of a nerve impulse
$10^{-3}$	Size of a grain of sand	$10^2$	Mass of a person	1	Time for one heartbeat
1	Height of a 4-year-old child	$10^3$	Mass of a car	$10^5$	One day ( $8.64 \times 10^4$ s)
$10^2$	Length of a football field	$10^8$	Mass of a large ship	$10^7$	One year (y) ( $3.16 \times 10^7$ s)
$10^4$	Greatest ocean depth	$10^{12}$	Mass of a large iceberg	$10^9$	About half the life expectancy of a human
$10^7$	Diameter of the Earth	$10^{15}$	Mass of the nucleus of a comet	$10^{11}$	Recorded history
$10^{11}$	Distance from the Earth to the Sun	$10^{23}$	Mass of the Moon ( $7.35 \times 10^{22}$ kg)	$10^{17}$	Age of the Earth
$10^{16}$	Distance traveled by light in 1 year (a light year)	$10^{25}$	Mass of the Earth ( $5.97 \times 10^{24}$ kg)	$10^{18}$	Age of the universe
$10^{21}$	Diameter of the Milky Way galaxy	$10^{30}$	Mass of the Sun ( $1.99 \times 10^{30}$ kg)		



Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
$10^{22}$	Distance from the Earth to the nearest large galaxy (Andromeda)	$10^{42}$	Mass of the Milky Way galaxy (current upper limit)		
$10^{26}$	Distance from the Earth to the edges of the known universe	$10^{53}$	Mass of the known universe (current upper limit)		

Approximate Values of Length, Mass, and Time

### Example:

#### Unit Conversions: A Short Drive Home

Suppose that you drive the 10.0 km from your university to home in 20.0 min. Calculate your average speed (a) in kilometers per hour (km/h) and (b) in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

#### Strategy

First we calculate the average speed using the given units. Then we can get the average speed into the desired units by picking the correct conversion factor and multiplying by it. The correct conversion factor is the one that cancels the unwanted unit and leaves the desired unit in its place.

#### Solution for (a)

(1) Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now—average speed and other motion concepts will be covered in a later module.) In equation form,

#### Equation:

$$\text{average speed} = \frac{\text{distance}}{\text{time}}.$$

(2) Substitute the given values for distance and time.

#### Equation:

$$\text{average speed} = \frac{10.0 \text{ km}}{20.0 \text{ min}} = 0.500 \frac{\text{km}}{\text{min}}.$$

(3) Convert km/min to km/h: multiply by the conversion factor that will cancel minutes and leave hours. That conversion factor is 60 min/hr. Thus,

#### Equation:

$$\text{average speed} = 0.500 \frac{\text{km}}{\text{min}} \times \frac{60 \text{ min}}{1 \text{ h}} = 30.0 \frac{\text{km}}{\text{h}}.$$

#### Discussion for (a)

To check your answer, consider the following:

(1) Be sure that you have properly cancelled the units in the unit conversion. If you have written the unit conversion factor upside down, the units will not cancel properly in the equation. If you accidentally get the ratio upside down, then the units will not cancel; rather, they will give you the wrong units as follows:

**Equation:**

$$\frac{\text{km}}{\text{min}} \times \frac{1 \text{ hr}}{60 \text{ min}} = \frac{1}{60} \frac{\text{km} \cdot \text{hr}}{\text{min}^2},$$

which are obviously not the desired units of km/h.

(2) Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of km/h and we have indeed obtained these units.

(3) Check the significant figures. Because each of the values given in the problem has three significant figures, the answer should also have three significant figures. The answer 30.0 km/hr does indeed have three significant figures, so this is appropriate. Note that the significant figures in the conversion factor are not relevant because an hour is *defined* to be 60 minutes, so the precision of the conversion factor is perfect.

(4) Next, check whether the answer is reasonable. Let us consider some information from the problem—if you travel 10 km in a third of an hour (20 min), you would travel three times that far in an hour. The answer does seem reasonable.

**Solution for (b)**

There are several ways to convert the average speed into meters per second.

(1) Start with the answer to (a) and convert km/h to m/s. Two conversion factors are needed—one to convert hours to seconds, and another to convert kilometers to meters.

(2) Multiplying by these yields

**Equation:**

$$\text{Average speed} = 30.0 \frac{\text{km}}{\text{h}} \times \frac{1 \text{ h}}{3,600 \text{ s}} \times \frac{1,000 \text{ m}}{1 \text{ km}},$$

**Equation:**

$$\text{Average speed} = 8.33 \frac{\text{m}}{\text{s}}.$$

**Discussion for (b)**

If we had started with 0.500 km/min, we would have needed different conversion factors, but the answer would have been the same: 8.33 m/s.

You may have noted that the answers in the worked example just covered were given to three digits.

Why? When do you need to be concerned about the number of digits in something you calculate? Why not write down all the digits your calculator produces? The module [Accuracy, Precision, and Significant Figures](#) will help you answer these questions.

**Note:**

**Nonstandard Units**

While there are numerous types of units that we are all familiar with, there are others that are much more obscure. For example, a **firkin** is a unit of volume that was once used to measure beer. One firkin equals about 34 liters. To learn more about nonstandard units, use a dictionary or encyclopedia to research different “weights and measures.” Take note of any unusual units, such as a barleycorn, that are not listed in the text. Think about how the unit is defined and state its relationship to SI units.

**Exercise:**  
**Check Your Understanding**

**Problem:**

Some hummingbirds beat their wings more than 50 times per second. A scientist is measuring the time it takes for a hummingbird to beat its wings once. Which fundamental unit should the scientist use to describe the measurement? Which factor of 10 is the scientist likely to use to describe the motion precisely? Identify the metric prefix that corresponds to this factor of 10.

---

**Solution:**

The scientist will measure the time between each movement using the fundamental unit of seconds. Because the wings beat so fast, the scientist will probably need to measure in milliseconds, or  $10^{-3}$  seconds. (50 beats per second corresponds to 20 milliseconds per beat.)

**Exercise:**  
**Check Your Understanding**

**Problem:**

One cubic centimeter is equal to one milliliter. What does this tell you about the different units in the SI metric system?

---

**Solution:**

The fundamental unit of length (meter) is probably used to create the derived unit of volume (liter). The measure of a milliliter is dependent on the measure of a centimeter.

**Summary**

- Physical quantities are a characteristic or property of an object that can be measured or calculated from other measurements.
- Units are standards for expressing and comparing the measurement of physical quantities. All units can be expressed as combinations of four fundamental units.
- The four fundamental units we will use in this text are the meter (for length), the kilogram (for mass), the second (for time), and the ampere (for electric current). These units are part of the metric system, which uses powers of 10 to relate quantities over the vast ranges encountered in nature.
- The four fundamental units are abbreviated as follows: meter, m; kilogram, kg; second, s; and ampere, A. The metric system also uses a standard set of prefixes to denote each order of magnitude greater than or lesser than the fundamental unit itself.
- Unit conversions involve changing a value expressed in one type of unit to another type of unit. This is done by using conversion factors, which are ratios relating equal quantities of different units.

**Conceptual Questions**

**Exercise:**

**Problem:** Identify some advantages of metric units.

**Problems & Exercises**

**Exercise:****Problem:**

The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?

---

**Solution:**

- a. 27.8 m/s
- b. 62.1 mph

**Exercise:****Problem:**

A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?

**Exercise:****Problem:**

Show that  $1.0 \text{ m/s} = 3.6 \text{ km/h}$ . Hint: Show the explicit steps involved in converting  $1.0 \text{ m/s} = 3.6 \text{ km/h}$ .

---

**Solution:**

$$\begin{aligned}\frac{1.0 \text{ m}}{\text{s}} &= \frac{1.0 \text{ m}}{\text{s}} \times \frac{3600 \text{ s}}{1 \text{ hr}} \times \frac{1 \text{ km}}{1000 \text{ m}} \\ &= 3.6 \text{ km/h.}\end{aligned}$$

**Exercise:****Problem:**

American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 meter equals 3.281 feet.)

**Exercise:****Problem:**

Soccer fields vary in size. A large soccer field is 115 m long and 85 m wide. What are its dimensions in feet and inches? (Assume that 1 meter equals 3.281 feet.)

---

**Solution:**

length: 377 ft;  $4.53 \times 10^3$  in. width: 280 ft;  $3.3 \times 10^3$  in.

**Exercise:****Problem:**

What is the height in meters of a person who is 6 ft 1.0 in. tall? (Assume that 1 meter equals 39.37 in.)

**Exercise:**

**Problem:**

Mount Everest, at 29,028 feet, is the tallest mountain on the Earth. What is its height in kilometers? (Assume that 1 kilometer equals 3,281 feet.)

---

**Solution:**

8.847 km

**Exercise:**

**Problem:** The speed of sound is measured to be 342 m/s on a certain day. What is this in km/h?

**Exercise:****Problem:**

Tectonic plates are large segments of the Earth's crust that move slowly. Suppose that one such plate has an average speed of 4.0 cm/year. (a) What distance does it move in 1 s at this speed? (b) What is its speed in kilometers per million years?

---

**Solution:**

(a)  $1.3 \times 10^{-9}$  m

(b) 40 km/My

**Exercise:****Problem:**

(a) Refer to [\[link\]](#) to determine the average distance between the Earth and the Sun. Then calculate the average speed of the Earth in its orbit in kilometers per second. (b) What is this in meters per second?

**Glossary**

physical quantity

a characteristic or property of an object that can be measured or calculated from other measurements

units

a standard used for expressing and comparing measurements

SI units

the international system of units that scientists in most countries have agreed to use; includes units such as meters, liters, and grams

English units

system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds

fundamental units

units that can only be expressed relative to the procedure used to measure them

derived units

units that can be calculated using algebraic combinations of the fundamental units

second

the SI unit for time, abbreviated (s)

meter

the SI unit for length, abbreviated (m)

kilogram

the SI unit for mass, abbreviated (kg)

metric system

a system in which values can be calculated in factors of 10

order of magnitude

refers to the size of a quantity as it relates to a power of 10

conversion factor

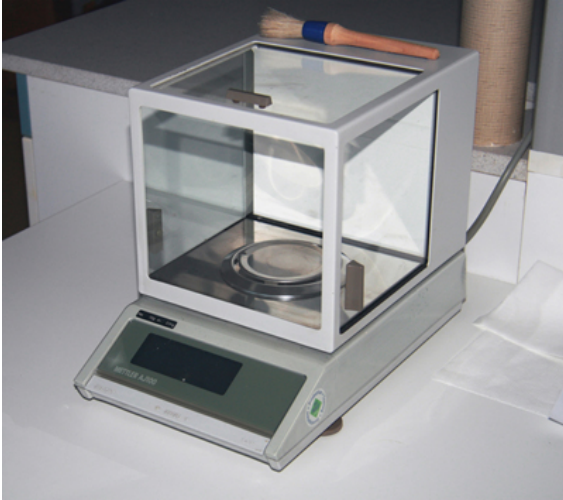
a ratio expressing how many of one unit are equal to another unit

## Accuracy, Precision, and Significant Figures

- Determine the appropriate number of significant figures in both addition and subtraction, as well as multiplication and division calculations.
- Calculate the percent uncertainty of a measurement.



A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The “known masses” are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams.  
(credit: Serge Melki)



Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. Whereas a mechanical balance may only read the mass of an object to the nearest tenth of a gram, many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit: Karel Jakubec)

## Accuracy and Precision of a Measurement

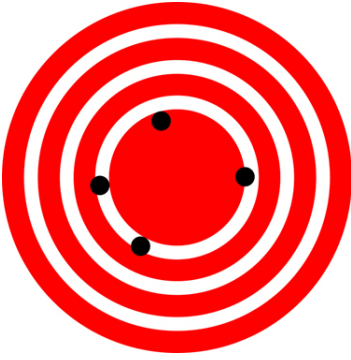
Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in.



These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9, then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In [\[link\]](#), you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low precision, high accuracy measuring system. However, in [\[link\]](#), the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.



A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy.  
(credit: Dark Evil)



In this figure,  
the dots are  
concentrated  
rather closely to  
one another,  
indicating high  
precision, but  
they are rather  
far away from  
the actual  
location of the  
restaurant,  
indicating low  
accuracy.  
(credit: Dark  
Evil)

## Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the

uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement,  $A$ , is often denoted as  $\delta A$  (“delta  $A$ ”), so the measurement result would be recorded as  $A \pm \delta A$ . In our paper example, the length of the paper could be expressed as  $11 \text{ in.} \pm 0.2$ .

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. The skill of the person making the measurement,
3. Irregularities in the object being measured,
4. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

**Note:**

**Making Connections: Real-World Connections – Fevers or Chills?**

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were  $3.0^\circ\text{C}$ ? If the child’s temperature reading was  $37.0^\circ\text{C}$  (which is normal body temperature), the “true” temperature could be anywhere from a

hypothermic 34.0°C to a dangerously high 40.0°C. A thermometer with an uncertainty of 3.0°C would be useless.

## Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement  $A$  is expressed with uncertainty,  $\delta A$ , the **percent uncertainty** (%unc) is defined to be

**Equation:**

$$\% \text{ unc} = \frac{\delta A}{A} \times 100\%.$$

### Example:

#### Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5-lb bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

Week 1 weight: 4.8 lb

Week 2 weight: 5.3 lb

Week 3 weight: 4.9 lb

Week 4 weight: 5.4 lb

You determine that the weight of the 5-lb bag has an uncertainty of  $\pm 0.4$  lb. What is the percent uncertainty of the bag's weight?

#### Strategy

First, observe that the expected value of the bag's weight,  $A$ , is 5 lb. The uncertainty in this value,  $\delta A$ , is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

**Equation:**

$$\% \text{ unc} = \frac{\delta A}{A} \times 100\%.$$

### Solution

Plug the known values into the equation:

### Equation:

$$\% \text{ unc} = \frac{0.4 \text{ lb}}{5 \text{ lb}} \times 100\% = 8\%.$$

### Discussion

We can conclude that the weight of the apple bag is  $5 \text{ lb} \pm 8\%$ . Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

## Uncertainties in Calculations

There is an uncertainty in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method says that *the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation*. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is  $12.0 \text{ m}^2$  and has an uncertainty of 3%. (Expressed as an area this is  $0.36 \text{ m}^2$ , which we round to  $0.4 \text{ m}^2$  since the area of the floor is given to a tenth of a square meter.)

### Exercise:

### Check Your Understanding

**Problem:**

A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of  $\pm 0.05$  s. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

---

**Solution:**

No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

**Precision of Measuring Tools and Significant Figures**

An important factor in the accuracy and precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter, while a caliper can measure length to the nearest 0.01 millimeter. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise and accurate the measurements can be.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be 36.7 cm. You could not express this value as 36.71 cm because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the

method of **significant figures**, the rule is that *the last digit written down in a measurement is the first digit with some uncertainty*. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

## Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placekeepers that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placekeepers but are significant—this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or they could be placekeepers. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) *Zeros are significant except when they serve only as placekeepers.*

### Exercise:

#### Check Your Understanding

##### Problem:

Determine the number of significant figures in the following measurements:

- a. 0.0009
- b. 15,450.0
- c.  $6 \times 10^3$
- d. 87.990
- e. 30.42

---

##### Solution:



- (a) 1; the zeros in this number are placekeepers that indicate the decimal point
- (b) 6; here, the zeros indicate that a measurement was made to the 0.1 decimal point, so the zeros are significant
- (c) 1; the value  $10^3$  signifies the decimal place, not the number of measured values
- (d) 5; the final zero indicates that a measurement was made to the 0.001 decimal point, so it is significant
- (e) 4; any zeros located in between significant figures in a number are also significant

## Significant Figures in Calculations

When combining measurements with different degrees of accuracy and precision, *the number of significant digits in the final answer can be no greater than the number of significant digits in the least precise measured value*. There are two different rules, one for multiplication and division and the other for addition and subtraction, as discussed below.

**1. For multiplication and division:** *The result should have the same number of significant figures as the quantity having the least significant figures entering into the calculation.* For example, the area of a circle can be calculated from its radius using  $A = \pi r^2$ . Let us see how many significant figures the area has if the radius has only two—say,  $r = 1.2$  m. Then,

**Equation:**

$$A = \pi r^2 = (3.1415927...) \times (1.2 \text{ m})^2 = 4.5238934 \text{ m}^2$$

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated

quantity to two significant figures or

**Equation:**

$$A=4.5 \text{ m}^2,$$

even though  $\pi$  is good to at least eight digits.

**2. For addition and subtraction:** *The answer can contain no more decimal places than the least precise measurement.* Suppose that you buy 7.56-kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg. Then you drop off 6.052-kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Finally, you go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do you now have, and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

**Equation:**

$$\begin{array}{r} 7.56 \text{ kg} \\ - 6.052 \text{ kg} \\ \hline +13.7 \text{ kg} \\ 15.208 \text{ kg} \end{array} = 15.2 \text{ kg}.$$

Next, we identify the least precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

### Significant Figures in this Text

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. You will note that an answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant

figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and more than three significant figures will be used. Finally, if a number is *exact*, such as the two in the formula for the circumference of a circle,  $c = 2\pi r$ , it does not affect the number of significant figures in a calculation.

**Exercise:**

**Check Your Understanding**

**Problem:**

Perform the following calculations and express your answer using the correct number of significant digits.

- (a) A woman has two bags weighing 13.5 pounds and one bag with a weight of 10.2 pounds. What is the total weight of the bags?
- (b) The force  $F$  on an object is equal to its mass  $m$  multiplied by its acceleration  $a$ . If a wagon with mass 55 kg accelerates at a rate of  $0.0255 \text{ m/s}^2$ , what is the force on the wagon? (The unit of force is called the newton, and it is expressed with the symbol N.)

---

**Solution:**

- (a) 37.2 pounds; Because the number of bags is an exact value, it is not considered in the significant figures.
- (b) 1.4 N; Because the value 55 kg has only two significant figures, the final value must also contain two significant figures.

**Note:**

**PhET Explorations: Estimation**

Explore size estimation in one, two, and three dimensions! Multiple levels of difficulty allow for progressive skill improvement.

[https://phet.colorado.edu/sims/estimation/estimation\\_en.html](https://phet.colorado.edu/sims/estimation/estimation_en.html)

## Summary

- Accuracy of a measured value refers to how close a measurement is to the correct value. The uncertainty in a measurement is an estimate of the amount by which the measurement result may differ from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements.
- The precision of a *measuring tool* is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.

## Conceptual Questions

### Exercise:

#### Problem:

What is the relationship between the accuracy and uncertainty of a measurement?

### Exercise:

#### Problem:

Prescriptions for vision correction are given in units called *diopters* (D). Determine the meaning of that unit. Obtain information (perhaps by calling an optometrist or performing an internet search) on the minimum uncertainty with which corrections in diopters are determined and the accuracy with which corrective lenses can be produced. Discuss the sources of uncertainties in both the prescription and accuracy in the manufacture of lenses.

## Problems & Exercises

Express your answers to problems in this section to the correct number of significant figures and proper units.

**Exercise:**

**Problem:**

Suppose that your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?

---

**Solution:**

2 kg

**Exercise:**

**Problem:**

A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?

**Exercise:**

**Problem:**

(a) A car speedometer has a 5.0% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) Convert this range to miles per hour. (1 km = 0.6214 mi)

---

**Solution:**

a. 85.5 to 94.5 km/h

b. 53.1 to 58.7 mi/h

**Exercise:**

**Problem:**

An infant's pulse rate is measured to be  $130 \pm 5$  beats/min. What is the percent uncertainty in this measurement?

**Exercise:****Problem:**

(a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 y? (b) In 2.00 y? (c) In 2.000 y?

---

**Solution:**

(a)  $7.6 \times 10^7$  beats

(b)  $7.57 \times 10^7$  beats

(c)  $7.57 \times 10^7$  beats

**Exercise:****Problem:**

A can contains 375 mL of soda. How much is left after 308 mL is removed?

**Exercise:****Problem:**

State how many significant figures are proper in the results of the following calculations: (a)  $(106.7)(98.2)/(46.210)(1.01)$  (b)  $(18.7)^2$  (c)  $(1.60 \times 10^{-19})(3712)$ .

---

**Solution:**

a. 3

b. 3

c. 3

**Exercise:**

**Problem:**

(a) How many significant figures are in the numbers 99 and 100? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers, significant figures or percent uncertainties?

**Exercise:****Problem:**

(a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/h, what is the range of speeds you could be going?

---

**Solution:**

a) 2.2%

(b) 59 to 61 km/h

**Exercise:****Problem:**

(a) A person's blood pressure is measured to be  $120 \pm 2$  mm Hg. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?

**Exercise:****Problem:**

A person measures his or her heart rate by counting the number of beats in 30 s. If  $40 \pm 1$  beats are counted in  $30.0 \pm 0.5$  s, what is the heart rate and its uncertainty in beats per minute?

---

**Solution:**

$80 \pm 3$  beats/min

**Exercise:**

**Problem:** What is the area of a circle 3.102 cm in diameter?

**Exercise:**

**Problem:**

If a marathon runner averages 9.5 mi/h, how long does it take him or her to run a 26.22-mi marathon?

---

**Solution:**

2.8 h

**Exercise:**

**Problem:**

A marathon runner completes a 42.188-km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?

**Exercise:**

**Problem:**

The sides of a small rectangular box are measured to be  $1.80 \pm 0.01$  cm,  $2.05 \pm 0.02$  cm, and  $3.1 \pm 0.1$  cm long. Calculate its volume and uncertainty in cubic centimeters.

---

**Solution:**

$11 \pm 1$  cm<sup>3</sup>

**Exercise:**



**Problem:**

When non-metric units were used in the United Kingdom, a unit of mass called the *pound-mass* (lbm) was employed, where  $1 \text{ lbm} = 0.4539 \text{ kg}$ . (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?

**Exercise:****Problem:**

The length and width of a rectangular room are measured to be  $3.955 \pm 0.005 \text{ m}$  and  $3.050 \pm 0.005 \text{ m}$ . Calculate the area of the room and its uncertainty in square meters.

---

**Solution:**

$$12.06 \pm 0.04 \text{ m}^2$$

**Exercise:****Problem:**

A car engine moves a piston with a circular cross section of  $7.500 \pm 0.002 \text{ cm}$  diameter a distance of  $3.250 \pm 0.001 \text{ cm}$  to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

**Glossary**

accuracy

the degree to which a measured value agrees with correct value for that measurement

method of adding percents

the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation

percent uncertainty

the ratio of the uncertainty of a measurement to the measured value, expressed as a percentage

precision

the degree to which repeated measurements agree with each other

significant figures

express the precision of a measuring tool used to measure a value

uncertainty

a quantitative measure of how much your measured values deviate from a standard or expected value

## Approximation

- Make reasonable approximations based on given data.

On many occasions, physicists, other scientists, and engineers need to make **approximations** or “guesstimates” for a particular quantity. What is the distance to a certain destination? What is the approximate density of a given item? About how large a current will there be in a circuit? Many approximate numbers are based on formulae in which the input quantities are known only to a limited accuracy. As you develop problem-solving skills (that can be applied to a variety of fields through a study of physics), you will also develop skills at approximating. You will develop these skills through thinking more quantitatively, and by being willing to take risks. As with any endeavor, experience helps, as well as familiarity with units. These approximations allow us to rule out certain scenarios or unrealistic numbers. Approximations also allow us to challenge others and guide us in our approaches to our scientific world. Let us do two examples to illustrate this concept.

### **Example:**

#### **Approximate the Height of a Building**

Can you approximate the height of one of the buildings on your campus, or in your neighborhood? Let us make an approximation based upon the height of a person. In this example, we will calculate the height of a 39-story building.

#### **Strategy**

Think about the average height of an adult male. We can approximate the height of the building by scaling up from the height of a person.

#### **Solution**

Based on information in the example, we know there are 39 stories in the building. If we use the fact that the height of one story is approximately equal to about the length of two adult humans (each human is about 2-m tall), then we can estimate the total height of the building to be

#### **Equation:**

$$\frac{2 \text{ m}}{1 \text{ person}} \times \frac{2 \text{ person}}{1 \text{ story}} \times 39 \text{ stories} = 156 \text{ m.}$$

### Discussion

You can use known quantities to determine an approximate measurement of unknown quantities. If your hand measures 10 cm across, how many hand lengths equal the width of your desk? What other measurements can you approximate besides length?

### Example:

#### Approximating Vast Numbers: a Trillion Dollars



A bank stack contains one-hundred \$100 bills, and is worth \$10,000. How many bank stacks make up a trillion dollars? (credit: Andrew Magill)

The U.S. federal deficit in the 2008 fiscal year was a little greater than \$10 trillion. Most of us do not have any concept of how much even one trillion actually is. Suppose that you were given a trillion dollars in \$100 bills. If you made 100-bill stacks and used them to evenly cover a football field (between the end zones), make an approximation of how high the money pile would become. (We will use feet/inches rather than meters here)

because football fields are measured in yards.) One of your friends says 3 in., while another says 10 ft. What do you think?

### **Strategy**

When you imagine the situation, you probably envision thousands of small stacks of 100 wrapped \$100 bills, such as you might see in movies or at a bank. Since this is an easy-to-approximate quantity, let us start there. We can find the volume of a stack of 100 bills, find out how many stacks make up one trillion dollars, and then set this volume equal to the area of the football field multiplied by the unknown height.

### **Solution**

(1) Calculate the volume of a stack of 100 bills. The dimensions of a single bill are approximately 3 in. by 6 in. A stack of 100 of these is about 0.5 in. thick. So the total volume of a stack of 100 bills is:

#### **Equation:**

$$\begin{aligned}\text{volume of stack} &= \text{length} \times \text{width} \times \text{height}, \\ \text{volume of stack} &= 6 \text{ in.} \times 3 \text{ in.} \times 0.5 \text{ in.}, \\ \text{volume of stack} &= 9 \text{ in.}^3.\end{aligned}$$

(2) Calculate the number of stacks. Note that a trillion dollars is equal to  $\$1 \times 10^{12}$ , and a stack of one-hundred \$100 bills is equal to \$10,000, or  $\$1 \times 10^4$ . The number of stacks you will have is:

#### **Equation:**

$$\$1 \times 10^{12} (\text{a trillion dollars}) / \$1 \times 10^4 \text{ per stack} = 1 \times 10^8 \text{ stacks.}$$

(3) Calculate the area of a football field in square inches. The area of a football field is 100 yd  $\times$  50 yd, which gives 5,000 yd<sup>2</sup>. Because we are working in inches, we need to convert square yards to square inches:

#### **Equation:**

$$\begin{aligned}\text{Area} &= 5,000 \text{ yd}^2 \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in.}}{1 \text{ ft}} \times \frac{12 \text{ in.}}{1 \text{ ft}} = 6,480,000 \text{ in.}^2, \\ \text{Area} &\approx 6 \times 10^6 \text{ in.}^2.\end{aligned}$$

This conversion gives us  $6 \times 10^6 \text{ in.}^2$  for the area of the field. (Note that we are using only one significant figure in these calculations.)

(4) Calculate the total volume of the bills. The volume of all the \$100-bill stacks is  $9 \text{ in.}^3/\text{stack} \times 10^8 \text{ stacks} = 9 \times 10^8 \text{ in.}^3$ .

(5) Calculate the height. To determine the height of the bills, use the equation:

**Equation:**

$$\text{volume of bills} = \text{area of field} \times \text{height of money:}$$

$$\text{Height of money} = \frac{\text{volume of bills}}{\text{area of field}},$$

$$\text{Height of money} = \frac{9 \times 10^8 \text{ in.}^3}{6 \times 10^6 \text{ in.}^2} = 1.33 \times 10^2 \text{ in.},$$

$$\text{Height of money} \approx 1 \times 10^2 \text{ in.} = 100 \text{ in.}$$

The height of the money will be about 100 in. high. Converting this value to feet gives

**Equation:**

$$100 \text{ in.} \times \frac{1 \text{ ft}}{12 \text{ in.}} = 8.33 \text{ ft} \approx 8 \text{ ft.}$$

### Discussion

The final approximate value is much higher than the early estimate of 3 in., but the other early estimate of 10 ft (120 in.) was roughly correct. How did the approximation measure up to your first guess? What can this exercise tell you in terms of rough “guesstimates” versus carefully calculated approximations?

### Exercise:

#### Check Your Understanding

##### Problem:

Using mental math and your understanding of fundamental units, approximate the area of a regulation basketball court. Describe the process you used to arrive at your final approximation.

---

##### Solution:

An average male is about two meters tall. It would take approximately 15 men laid out end to end to cover the length, and about 7 to cover the width. That gives an approximate area of  $420 \text{ m}^2$ .

## Summary

Scientists often approximate the values of quantities to perform calculations and analyze systems.

## Problems & Exercises

### Exercise:

**Problem:** How many heartbeats are there in a lifetime?

---

### Solution:

Sample answer:  $2 \times 10^9$  heartbeats

### Exercise:

### Problem:

A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?

### Exercise:

### Problem:

How many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human? (Hint: The lifetime of an unstable atomic nucleus is on the order of  $10^{-22} \text{ s}$ .)

---

### Solution:

Sample answer:  $2 \times 10^{31}$  if an average human lifetime is taken to be about 70 years.

## Exercise:

### Problem:

Calculate the approximate number of atoms in a bacterium. Assume that the average mass of an atom in the bacterium is ten times the mass of a hydrogen atom. (Hint: The mass of a hydrogen atom is on the order of  $10^{-27}$  kg and the mass of a bacterium is on the order of  $10^{-15}$  kg.)



This color-enhanced photo shows *Salmonella typhimurium* (red) attacking human cells. These bacteria are commonly known for causing foodborne illness. Can you estimate the number of atoms in each bacterium? (credit: Rocky Mountain Laboratories, NIAID, NIH)

## Exercise:



**Problem:**

Approximately how many atoms thick is a cell membrane, assuming all atoms there average about twice the size of a hydrogen atom?

---

**Solution:**

Sample answer: 50 atoms

**Exercise:****Problem:**

(a) What fraction of Earth's diameter is the greatest ocean depth? (b) The greatest mountain height?

**Exercise:****Problem:**

(a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is ten times the mass of a bacterium. (b) Making the same assumption, how many cells are there in a human?

---

**Solution:**

Sample answers:

(a)  $10^{12}$  cells/hummingbird

(b)  $10^{16}$  cells/human

**Exercise:****Problem:**

Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?

**Glossary**

approximation

an estimated value based on prior experience and reasoning

## Introduction to One-Dimensional Kinematics

class="introduction"

The motion  
of an  
American  
kestrel  
through the  
air can be  
described by  
the bird's  
displacement  
, speed,  
velocity, and  
acceleration.  
When it flies  
in a straight  
line without  
any change  
in direction,  
its motion is  
said to be  
one  
dimensional.  
(credit: Vince  
Maidens,  
Wikimedia  
Commons)



Objects are in motion everywhere we look. Everything from a tennis game to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. And even in inanimate objects, there is continuous motion in the vibrations of atoms and molecules. Questions about motion are interesting in and of themselves: *How long will it take for a space probe to get to Mars? Where will a football land if it is thrown at a certain angle?* But an understanding of motion is also key to understanding other concepts in physics. An understanding of acceleration, for example, is crucial to the study of force.

Our formal study of physics begins with **kinematics** which is defined as the *study of motion without considering its causes*. The word “kinematics” comes from a Greek term meaning motion and is related to other English words such as “cinema” (movies) and “kinesiology” (the study of human motion). In one-dimensional kinematics and [Two-Dimensional Kinematics](#) we will study only the *motion* of a football, for example, without worrying about what forces cause or change its motion. Such considerations come in other chapters. In this chapter, we examine the simplest type of motion—namely, motion along a straight line, or one-dimensional motion. In [Two-Dimensional Kinematics](#), we apply concepts developed here to study motion along curved paths (two- and three-dimensional motion); for example, that of a car rounding a curve.

## Displacement

- Define position, displacement, distance, and distance traveled.
- Explain the relationship between position and displacement.
- Distinguish between displacement and distance traveled.
- Calculate displacement and distance given initial position, final position, and the path between the two.



These cyclists in Vietnam can be described by their position relative to buildings and a canal. Their motion can be described by their change in position, or displacement, in the frame of reference. (credit: Suzan Black, Fotopedia)

## Position

In order to describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For

example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See [\[link\]](#).) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See [\[link\]](#).)

## Displacement

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement**. The word “displacement” implies that an object has moved, or has been displaced.

### Note:

#### Displacement

Displacement is the *change in position* of an object:

#### Equation:

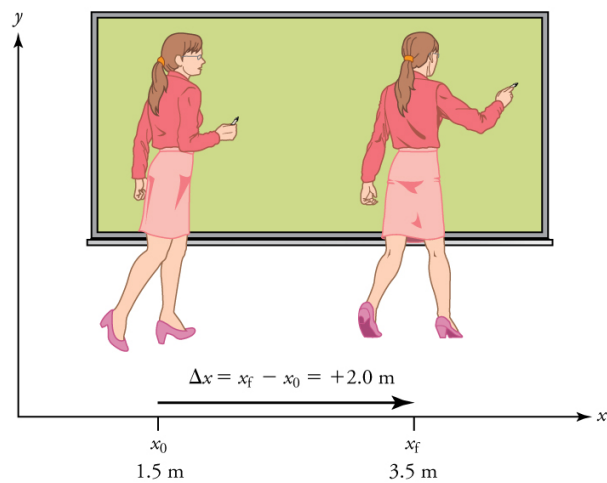
$$\Delta x = x_f - x_0,$$

where  $\Delta x$  is displacement,  $x_f$  is the final position, and  $x_0$  is the initial position.

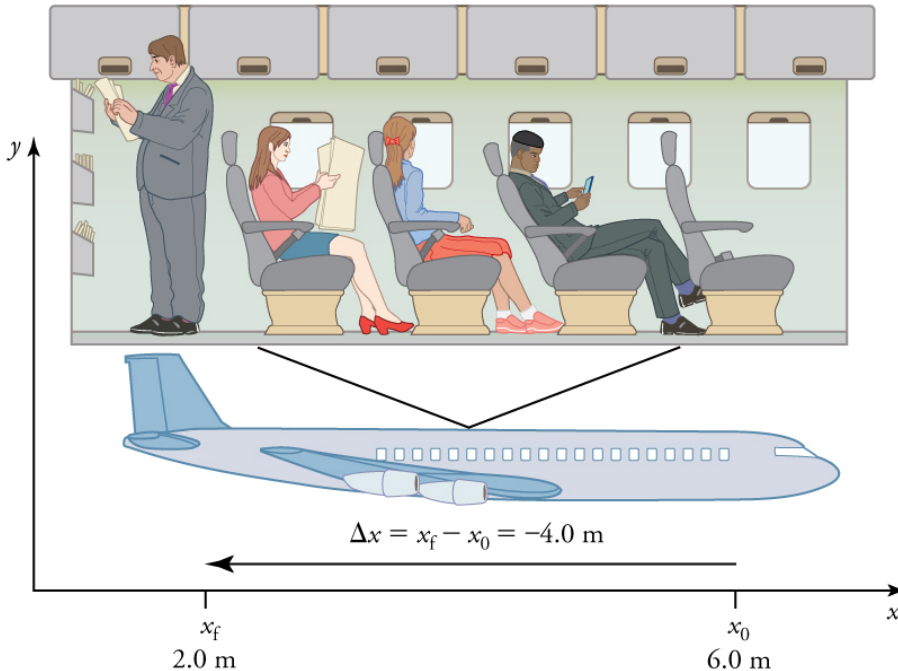
In this text the upper case Greek letter  $\Delta$  (delta) always means “change in” whatever quantity follows it; thus,  $\Delta x$  means *change in position*. Always solve for displacement by subtracting initial position  $x_0$  from final position  $x_f$ .

Note that the SI unit for displacement is the meter (m) (see [Physical Quantities and Units](#)), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are

used in a problem, you may need to convert them into meters to complete the calculation.



A professor paces left and right while lecturing. Her position relative to Earth is given by  $x$ . The  $+2.0 \text{ m}$  displacement of the professor relative to Earth is represented by an arrow pointing to the right.



A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by  $x$ . The  $-4.0\text{-m}$  displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in [\[link\]](#).

Note that displacement has a direction as well as a magnitude. The professor's displacement is  $2.0 \text{ m}$  to the right, and the airline passenger's displacement is  $4.0 \text{ m}$  toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is  $x_0 = 1.5 \text{ m}$  and her final position is  $x_f = 3.5 \text{ m}$ . Thus her displacement is

**Equation:**

$$\Delta x = x_f - x_0 = 3.5 \text{ m} - 1.5 \text{ m} = +2.0 \text{ m}.$$



In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is  $x_0 = 6.0$  m and his final position is  $x_f = 2.0$  m, so his displacement is

**Equation:**

$$\Delta x = x_f - x_0 = 2.0 \text{ m} - 6.0 \text{ m} = -4.0 \text{ m}.$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative  $x$  direction in our coordinate system.

## Distance

Although displacement is described in terms of direction, distance is not. **Distance** is defined to be *the magnitude or size of displacement between two positions*. Note that the distance between two positions is not the same as the distance traveled between them. **Distance traveled** is *the total length of the path traveled between two positions*. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

### Note:

#### Misconception Alert: Distance Traveled vs. Magnitude of Displacement

It is important to note that the *distance traveled*, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we nearly always deal with displacement and magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The

displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

### Exercise:

#### Check Your Understanding

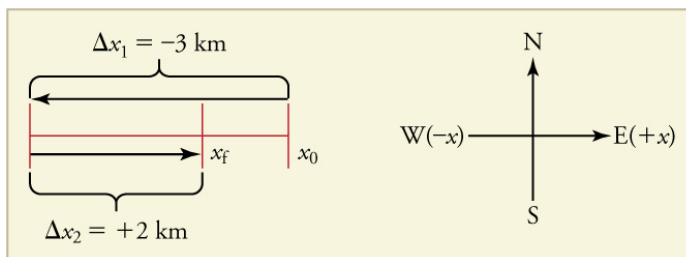
##### Problem:

A cyclist rides 3 km west and then turns around and rides 2 km east.

(a) What is her displacement? (b) What distance does she ride? (c)

What is the magnitude of her displacement?

##### Solution:



(a) The rider's displacement is  $\Delta x = x_f - x_0 = -1 \text{ km}$ . (The displacement is negative because we take east to be positive and west to be negative.)

(b) The distance traveled is  $3 \text{ km} + 2 \text{ km} = 5 \text{ km}$ .

(c) The magnitude of the displacement is  $1 \text{ km}$ .

### Section Summary

- Kinematics is the study of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object.

- In symbols, displacement  $\Delta x$  is defined to be  
**Equation:**

$$\Delta x = x_f - x_0,$$

where  $x_0$  is the initial position and  $x_f$  is the final position. In this text, the Greek letter  $\Delta$  (delta) always means “change in” whatever quantity follows it. The SI unit for displacement is the meter (m). Displacement has a direction as well as a magnitude.

- When you start a problem, assign which direction will be positive.
- Distance is the magnitude of displacement between two positions.
- Distance traveled is the total length of the path traveled between two positions.

## Conceptual Questions

### Exercise:

#### Problem:

Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Specifically identify each quantity in your example.

### Exercise:

#### Problem:

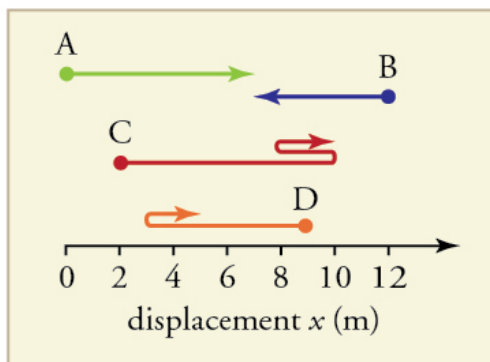
Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the same?

### Exercise:

#### Problem:

Bacteria move back and forth by using their flagella (structures that look like little tails). Speeds of up to  $50 \mu\text{m/s}$   $50 \times 10^{-6} \text{ m/s}$  have been observed. The total distance traveled by a bacterium is large for its size, while its displacement is small. Why is this?

## Problems & Exercises



### Exercise:

#### Problem:

Find the following for path A in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

---

#### Solution:

(a) 7 m

(b) 7 m

(c) +7 m

### Exercise:

#### Problem:

Find the following for path B in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

### Exercise:

**Problem:**

Find the following for path C in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

---

**Solution:**

(a) 13 m

(b) 9 m

(c) +9 m

**Exercise:****Problem:**

Find the following for path D in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

**Glossary**

kinematics

the study of motion without considering its causes

position

the location of an object at a particular time

displacement

the change in position of an object

distance

the magnitude of displacement between two positions

distance traveled

the total length of the path traveled between two positions

## Vectors, Scalars, and Coordinate Systems

- Define and distinguish between scalar and vector quantities.
- Assign a coordinate system for a scenario involving one-dimensional motion.



The motion of this Eclipse Concept jet can be described in terms of the distance it has traveled (a scalar quantity) or its displacement in a specific direction (a vector quantity). In order to specify the direction of motion, its displacement must be described based on a coordinate system. In this case, it may be convenient to choose motion toward the left as positive motion (it is the forward direction for the plane), although in many cases, the  $x$ -coordinate runs from left to right, with motion to the right as positive and motion to the left as negative. (credit: Armchair Aviator, Flickr)

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with both *magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

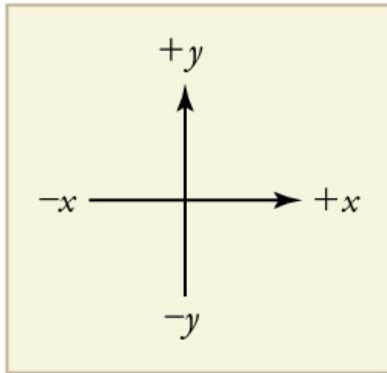
The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar** is any quantity that has a magnitude, but no direction. For example, a 20°C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars—quantities with no specified direction. Note, however, that a scalar can be negative, such as a −20°C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

## Coordinate Systems for One-Dimensional Motion

In order to describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive and motion down is negative. In some cases, however, as with the jet in [\[link\]](#), it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are

running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.



It is usually convenient to consider motion upward or to the right as positive (+) and motion downward or to the left as negative (−).

### **Exercise:**

#### **Check Your Understanding**

##### **Problem:**

A person's speed can stay the same as he or she rounds a corner and changes direction. Given this information, is speed a scalar or a vector quantity? Explain.

---

##### **Solution:**



Speed is a scalar quantity. It does not change at all with direction changes; therefore, it has magnitude only. If it were a vector quantity, it would change as direction changes (even if its magnitude remained constant).

## Section Summary

- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

## Conceptual Questions

### Exercise:

#### Problem:

A student writes, “A bird that is diving for prey has a speed of  $-10\text{ m/s}$ .” What is wrong with the student’s statement? What has the student actually described? Explain.

### Exercise:

**Problem:** What is the speed of the bird in [\[link\]](#)?

### Exercise:

#### Problem:

Acceleration is the change in velocity over time. Given this information, is acceleration a vector or a scalar quantity? Explain.

### Exercise:

**Problem:**

A weather forecast states that the temperature is predicted to be  $-5^{\circ}\text{C}$  the following day. Is this temperature a vector or a scalar quantity? Explain.

**Glossary**

scalar

a quantity that is described by magnitude, but not direction

vector

a quantity that is described by both magnitude and direction

## Time, Velocity, and Speed

- Explain the relationships between instantaneous velocity, average velocity, instantaneous speed, average speed, displacement, and time.
- Calculate velocity and speed given initial position, initial time, final position, and final time.
- Derive a graph of velocity vs. time given a graph of position vs. time.
- Interpret a graph of velocity vs. time.



The motion of these racing snails can be described by their speeds and their velocities.  
(credit: tobiasflickr, Flickr)

There is more to motion than distance and displacement. Questions such as, “How long does a foot race take?” and “What was the runner’s speed?” cannot be answered without an understanding of other concepts. In this section we add definitions of time, velocity, and speed to expand our description of motion.

## Time

As discussed in [Physical Quantities and Units](#), the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in

some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple—**time** is *change*, or the interval over which change occurs. It is impossible to know that time has passed unless something changes.

The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. **Elapsed time**  $\Delta t$  is the difference between the ending time and beginning time,

**Equation:**

$$\Delta t = t_f - t_0,$$

where  $\Delta t$  is the change in time or elapsed time,  $t_f$  is the time at the end of the motion, and  $t_0$  is the time at the beginning of the motion. (As usual, the delta symbol,  $\Delta$ , means the change in the quantity that follows it.)

Life is simpler if the beginning time  $t_0$  is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If  $t_0 = 0$ , then  $\Delta t = t_f \equiv t$ .

In this text, for simplicity's sake,

- motion starts at time equal to zero ( $t_0 = 0$ )
- the symbol  $t$  is used for elapsed time unless otherwise specified ( $\Delta t = t_f \equiv t$ )

## Velocity

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

### Note:

#### Average Velocity

**Average velocity** is *displacement (change in position) divided by the time of travel*,

#### Equation:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0},$$

where  $\bar{v}$  is the *average* (indicated by the bar over the  $v$ ) velocity,  $\Delta x$  is the change in position (or displacement), and  $x_f$  and  $x_0$  are the final and beginning positions at times  $t_f$  and  $t_0$ , respectively. If the starting time  $t_0$  is taken to be zero, then the average velocity is simply

#### Equation:

$$\bar{v} = \frac{\Delta x}{t}.$$

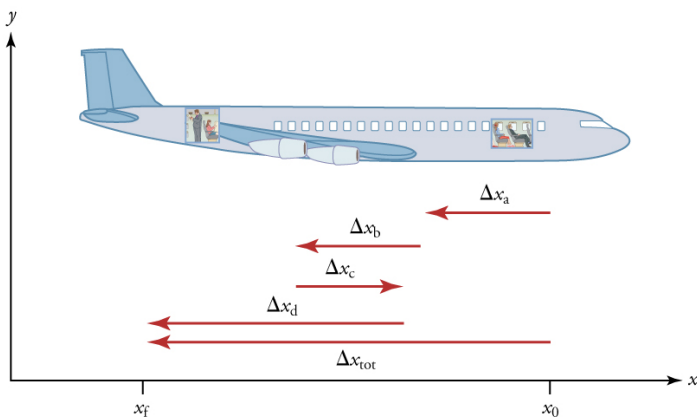
Notice that this definition indicates that *velocity is a vector because displacement is a vector*. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move  $-4$  m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be

#### Equation:

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}.$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.



A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the *instantaneous velocity* or the *velocity at a specific instant*. A car's speedometer, for example, shows the magnitude (but not the

direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.)

**Instantaneous velocity**  $v$  is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity,  $v$ , at a precise instant  $t$  can involve taking a limit, a calculus operation beyond the scope of this text.

However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

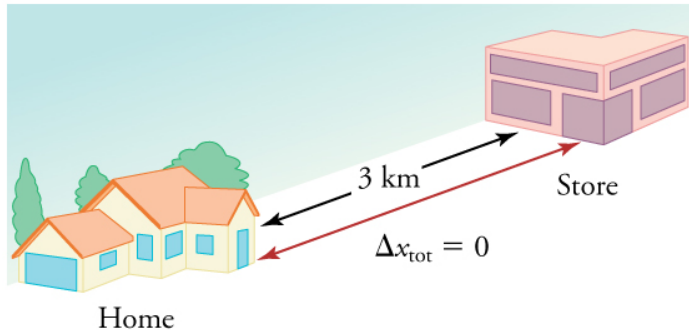
## Speed

In everyday language, most people use the terms “speed” and “velocity” interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus *speed is a scalar*. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

**Instantaneous speed** is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of  $-3.0$  m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was  $3.0$  m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is  $40$  km/h due north. Your instantaneous speed at that instant would be  $40$  km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car’s odometer shows the total distance traveled was  $6$  km, then your average speed was  $12$  km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero.

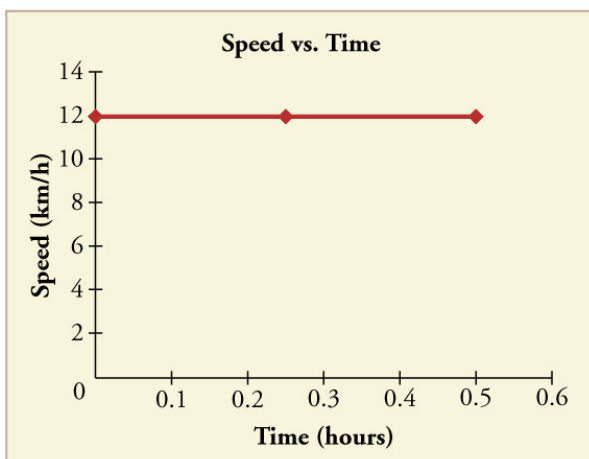
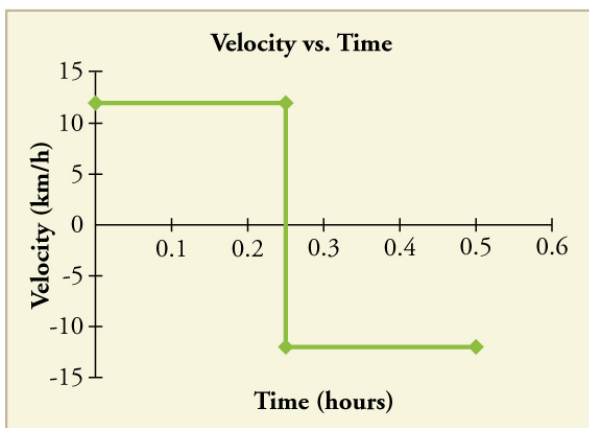
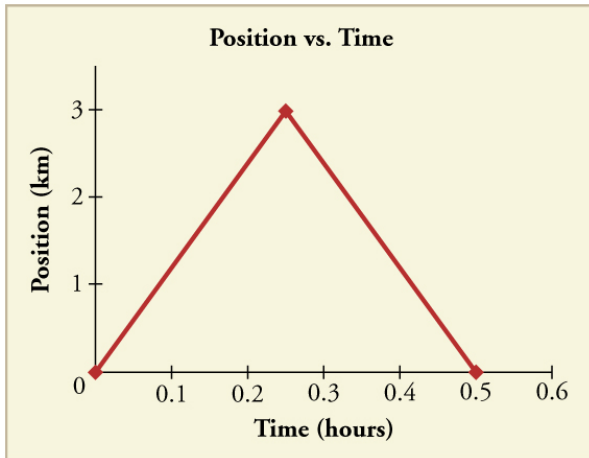
(Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is *not* simply the magnitude of average velocity.



During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in [\[link\]](#). (Note that these graphs depict a very simplified **model** of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we'll probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)





Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.

**Note:****Making Connections: Take-Home Investigation—Getting a Sense of Speed**

If you have spent much time driving, you probably have a good sense of speeds between about 10 and 70 miles per hour. But what are these in meters per second? What do we mean when we say that something is moving at 10 m/s? To get a better sense of what these values really mean, do some observations and calculations on your own:

- calculate typical car speeds in meters per second
- estimate jogging and walking speed by timing yourself; convert the measurements into both m/s and mi/h
- determine the speed of an ant, snail, or falling leaf

**Exercise:****Check Your Understanding****Problem:**

A commuter train travels from Baltimore to Washington, DC, and back in 1 hour and 45 minutes. The distance between the two stations is approximately 40 miles. What is (a) the average velocity of the train, and (b) the average speed of the train in m/s?

**Solution:**

(a) The average velocity of the train is zero because  $x_f = x_0$ ; the train ends up at the same place it starts.

(b) The average speed of the train is calculated below. Note that the train travels 40 miles one way and 40 miles back, for a total distance of 80 miles.

**Equation:**

$$\frac{\text{distance}}{\text{time}} = \frac{80 \text{ miles}}{105 \text{ minutes}}$$

**Equation:**

$$\frac{80 \text{ miles}}{105 \text{ minutes}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{1 \text{ meter}}{3.28 \text{ feet}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 20 \text{ m/s}$$

## Section Summary

- Time is measured in terms of change, and its SI unit is the second (s). Elapsed time for an event is

**Equation:**

$$\Delta t = t_f - t_0,$$

where  $t_f$  is the final time and  $t_0$  is the initial time. The initial time is often taken to be zero, as if measured with a stopwatch; the elapsed time is then just  $t$ .

- Average velocity  $\bar{v}$  is defined as displacement divided by the travel time. In symbols, average velocity is

**Equation:**

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}.$$

- The SI unit for velocity is m/s.
- Velocity is a vector and thus has a direction.
- Instantaneous velocity  $v$  is the velocity at a specific instant or the average velocity for an infinitesimal interval.
- Instantaneous speed is the magnitude of the instantaneous velocity.
- Instantaneous speed is a scalar quantity, as it has no direction specified.
- Average speed is the total distance traveled divided by the elapsed time. (Average speed is *not* the magnitude of the average velocity.) Speed is a scalar quantity; it has no direction associated with it.

## Conceptual Questions

**Exercise:****Problem:**

Give an example (but not one from the text) of a device used to measure time and identify what change in that device indicates a change in time.

**Exercise:****Problem:**

There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.

**Exercise:****Problem:**

Does a car's odometer measure position or displacement? Does its speedometer measure speed or velocity?

**Exercise:****Problem:**

If you divide the total distance traveled on a car trip (as determined by the odometer) by the time for the trip, are you calculating the average speed or the magnitude of the average velocity? Under what circumstances are these two quantities the same?

**Exercise:****Problem:**

How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

**Problems & Exercises****Exercise:**

**Problem:**

(a) Calculate Earth's average speed relative to the Sun. (b) What is its average velocity over a period of one year?

---

**Solution:**

(a)  $3.0 \times 10^4 \text{ m/s}$

(b) 0 m/s

**Exercise:****Problem:**

A helicopter blade spins at exactly 100 revolutions per minute. Its tip is 5.00 m from the center of rotation. (a) Calculate the average speed of the blade tip in the helicopter's frame of reference. (b) What is its average velocity over one revolution?

**Exercise:****Problem:**

The North American and European continents are moving apart at a rate of about 3 cm/y. At this rate how long will it take them to drift 500 km farther apart than they are at present?

---

**Solution:**

$2 \times 10^7$  years

**Exercise:**

**Problem:**

Land west of the San Andreas fault in southern California is moving at an average velocity of about 6 cm/y northwest relative to land east of the fault. Los Angeles is west of the fault and may thus someday be at the same latitude as San Francisco, which is east of the fault. How far in the future will this occur if the displacement to be made is 590 km northwest, assuming the motion remains constant?

**Exercise:****Problem:**

On May 26, 1934, a streamlined, stainless steel diesel train called the Zephyr set the world's nonstop long-distance speed record for trains. Its run from Denver to Chicago took 13 hours, 4 minutes, 58 seconds, and was witnessed by more than a million people along the route. The total distance traveled was 1633.8 km. What was its average speed in km/h and m/s?

---

**Solution:**

$$34.689 \text{ m/s} = 124.88 \text{ km/h}$$

**Exercise:****Problem:**

Tidal friction is slowing the rotation of the Earth. As a result, the orbit of the Moon is increasing in radius at a rate of approximately 4 cm/year. Assuming this to be a constant rate, how many years will pass before the radius of the Moon's orbit increases by  $3.84 \times 10^6 \text{ m}$  (1%)?

**Exercise:**

**Problem:**

A student drove to the university from her home and noted that the odometer reading of her car increased by 12.0 km. The trip took 18.0 min. (a) What was her average speed? (b) If the straight-line distance from her home to the university is 10.3 km in a direction  $25.0^\circ$  south of east, what was her average velocity? (c) If she returned home by the same path 7 h 30 min after she left, what were her average speed and velocity for the entire trip?

---

**Solution:**

(a) 40.0 km/h

(b) 34.3 km/h,  $25^\circ$  S of E.

(c) average speed = 3.20 km/h,  $\bar{v} = 0$ .

**Exercise:****Problem:**

The speed of propagation of the action potential (an electrical signal) in a nerve cell depends (inversely) on the diameter of the axon (nerve fiber). If the nerve cell connecting the spinal cord to your feet is 1.1 m long, and the nerve impulse speed is 18 m/s, how long does it take for the nerve signal to travel this distance?

**Exercise:**

**Problem:**

Conversations with astronauts on the lunar surface were characterized by a kind of echo in which the earthbound person's voice was so loud in the astronaut's space helmet that it was picked up by the astronaut's microphone and transmitted back to Earth. It is reasonable to assume that the echo time equals the time necessary for the radio wave to travel from the Earth to the Moon and back (that is, neglecting any time delays in the electronic equipment). Calculate the distance from Earth to the Moon given that the echo time was 2.56 s and that radio waves travel at the speed of light ( $3.00 \times 10^8$  m/s).

---

**Solution:**

384,000 km

**Exercise:****Problem:**

A football quarterback runs 15.0 m straight down the playing field in 2.50 s. He is then hit and pushed 3.00 m straight backward in 1.75 s. He breaks the tackle and runs straight forward another 21.0 m in 5.20 s. Calculate his average velocity (a) for each of the three intervals and (b) for the entire motion.

**Exercise:****Problem:**

The planetary model of the atom pictures electrons orbiting the atomic nucleus much as planets orbit the Sun. In this model you can view hydrogen, the simplest atom, as having a single electron in a circular orbit  $1.06 \times 10^{-10}$  m in diameter. (a) If the average speed of the electron in this orbit is known to be  $2.20 \times 10^6$  m/s, calculate the number of revolutions per second it makes about the nucleus. (b) What is the electron's average velocity?

---

**Solution:**



(a)  $6.61 \times 10^{15} \text{ rev/s}$

(b) 0 m/s

## Glossary

average speed

distance traveled divided by time during which motion occurs

average velocity

displacement divided by time over which displacement occurs

instantaneous velocity

velocity at a specific instant, or the average velocity over an infinitesimal time interval

instantaneous speed

magnitude of the instantaneous velocity

time

change, or the interval over which change occurs

model

simplified description that contains only those elements necessary to describe the physics of a physical situation

elapsed time

the difference between the ending time and beginning time

## Acceleration

- Define and distinguish between instantaneous acceleration, average acceleration, and deceleration.
- Calculate acceleration given initial time, initial velocity, final time, and final velocity.



A plane decelerates, or slows down, as it comes in for landing in St. Maarten. Its acceleration is opposite in direction to its velocity. (credit: Steve Conry, Flickr)

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the **acceleration**, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

### **Note:**

#### **Average Acceleration**

**Average Acceleration** is *the rate at which velocity changes*,

#### **Equation:**

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0},$$

where  $\bar{a}$  is average acceleration,  $v$  is velocity, and  $t$  is time. (The bar over the  $a$  means *average* acceleration.)

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are  $\text{m/s}^2$ , meters per second squared or meters per second per second, which literally means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in *direction*. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

**Note:**

**Acceleration as a Vector**

Acceleration is a vector in the same direction as the *change* in velocity,  $\Delta v$ . Since velocity is a vector, it can change either in magnitude or in direction. Acceleration is therefore a change in either speed or direction, or both.

Keep in mind that although acceleration is in the direction of the *change* in velocity, it is not always in the direction of *motion*. When an object slows down, its acceleration is opposite to the direction of its motion. This is known as **deceleration**.

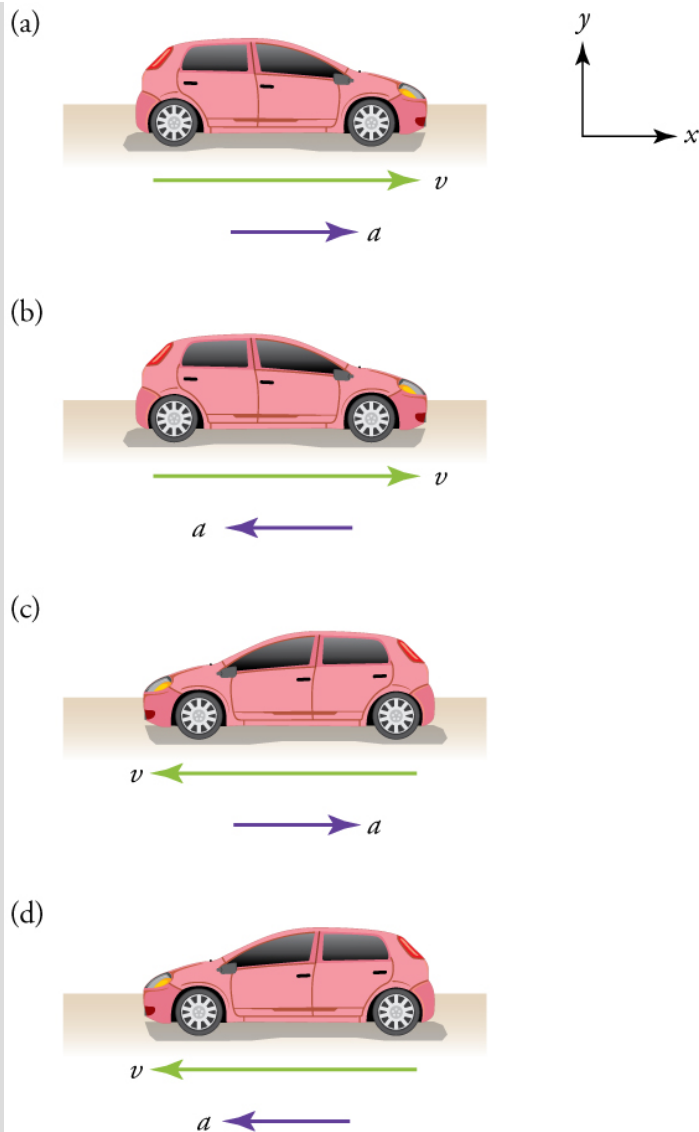


A subway train in Sao Paulo, Brazil, decelerates as it comes into a station. It is accelerating in a direction opposite to its direction of motion. (credit: Yusuke Kawasaki, Flickr)

**Note:**

**Misconception Alert: Deceleration vs. Negative Acceleration**

Deceleration always refers to acceleration in the direction opposite to the direction of the velocity. Deceleration always reduces speed. Negative acceleration, however, is acceleration *in the negative direction in the chosen coordinate system*. Negative acceleration may or may not be deceleration, and deceleration may or may not be considered negative acceleration. For example, consider [\[link\]](#).



(a) This car is speeding up as it moves toward the right. It therefore has positive acceleration in our coordinate system. (b) This car is slowing down as it moves toward the right. Therefore, it has negative acceleration in our coordinate system, because its acceleration is toward the left. The car is also decelerating: the direction of its acceleration is opposite to its direction of motion. (c) This car is moving

toward the left, but slowing down over time. Therefore, its acceleration is positive in our coordinate system because it is toward the right.

However, the car is decelerating because its acceleration is opposite to its motion. (d) This car is speeding up as it moves toward the left. It has negative acceleration because it is accelerating toward the left. However, because its acceleration is in the same direction as its motion, it is speeding up (*not* decelerating).

### **Example:**

#### **Calculating Acceleration: A Racehorse Leaves the Gate**

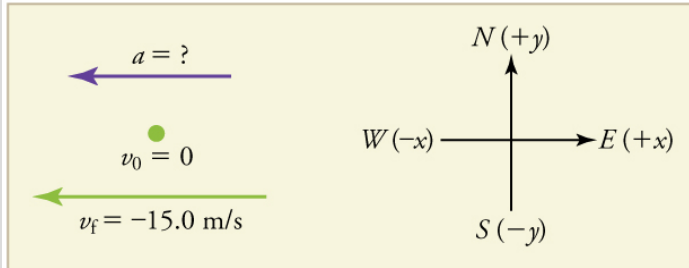
A racehorse coming out of the gate accelerates from rest to a velocity of  $15.0 \text{ m/s}$  due west in  $1.80 \text{ s}$ . What is its average acceleration?



(credit: Jon Sullivan, PD  
Photo.org)

### Strategy

First we draw a sketch and assign a coordinate system to the problem. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.



We can solve this problem by identifying  $\Delta v$  and  $\Delta t$  from the given information and then calculating the average acceleration directly from the equation  $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$ .

### Solution

1. Identify the knowns.  $v_0 = 0$ ,  $v_f = -15.0 \text{ m/s}$  (the negative sign indicates direction toward the west),  $\Delta t = 1.80 \text{ s}$ .
2. Find the change in velocity. Since the horse is going from zero to  $-15.0 \text{ m/s}$ , its change in velocity equals its final velocity:

$$\Delta v = v_f = -15.0 \text{ m/s}.$$

3. Plug in the known values ( $\Delta v$  and  $\Delta t$ ) and solve for the unknown  $\bar{a}$ .

### Equation:

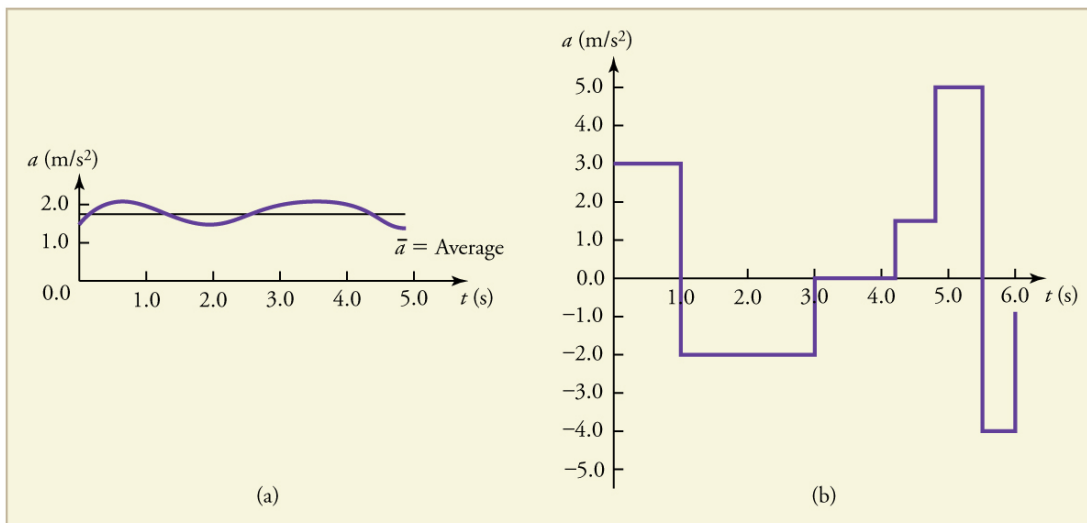
$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0 \text{ m/s}}{1.80 \text{ s}} = -8.33 \text{ m/s}^2.$$

### Discussion

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of  $8.33 \text{ m/s}^2$  due west means that the horse increases its velocity by  $8.33 \text{ m/s}$  due west each second, that is,  $8.33$  meters per second per second, which we write as  $8.33 \text{ m/s}^2$ . This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

## Instantaneous Acceleration

**Instantaneous acceleration**  $a$ , or the *acceleration at a specific instant in time*, is obtained by the same process as discussed for instantaneous velocity in [Time, Velocity, and Speed](#)—that is, by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. [\[link\]](#) shows graphs of instantaneous acceleration versus time for two very different motions. In [\[link\]](#)(a), the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In [\[link\]](#)(b), the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of  $+3.0 \text{ m/s}^2$  and  $-2.0 \text{ m/s}^2$ , respectively.

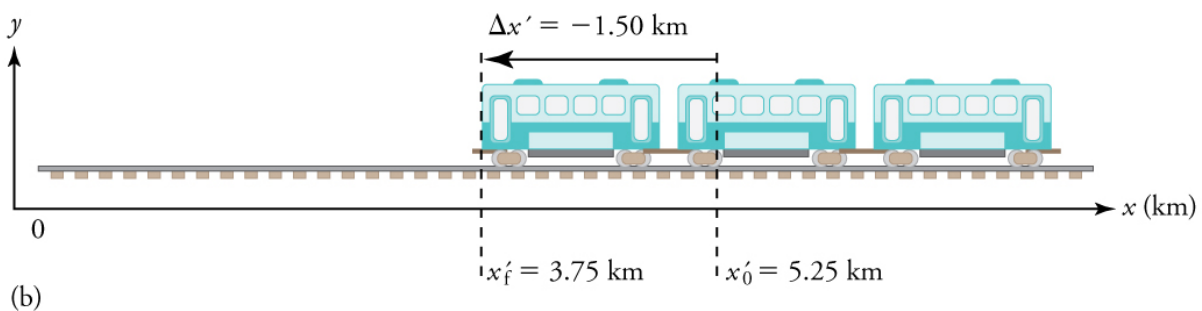
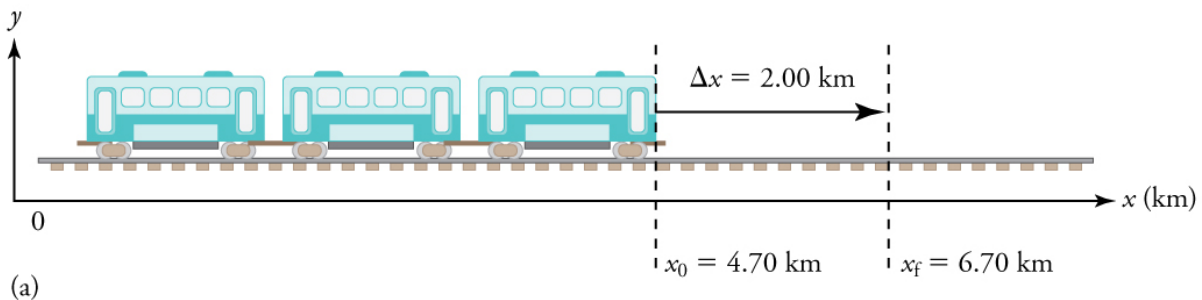


Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the



acceleration at any given time. (b) Here the acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

The next several examples consider the motion of the subway train shown in [\[link\]](#). In (a) the shuttle moves to the right, and in (b) it moves to the left. The examples are designed to further illustrate aspects of motion and to illustrate some of the reasoning that goes into solving problems.



One-dimensional motion of a subway train considered in [\[link\]](#), [\[link\]](#), [\[link\]](#), [\[link\]](#), [\[link\]](#), and [\[link\]](#). Here we have chosen the  $x$ -axis so that  $+$  means to the right and  $-$  means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from  $x_0$  to  $x_f$ . Its displacement  $\Delta x$  is  $+2.0$  km. (b) The train moves to the left from  $x'_0$  to  $x'_f$ . Its displacement  $\Delta x'$  is

–1.5 km. (Note that the prime symbol (') is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.)

**Example:****Calculating Displacement: A Subway Train**

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of [\[link\]](#)?

**Strategy**

A drawing with a coordinate system is already provided, so we don't need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation  $\Delta x = x_f - x_0$ . This is straightforward since the initial and final positions are given.

**Solution**

1. Identify the knowns. In the figure we see that  $x_f = 6.70$  km and  $x_0 = 4.70$  km for part (a), and  $x'_f = 3.75$  km and  $x'_0 = 5.25$  km for part (b).
2. Solve for displacement in part (a).

**Equation:**

$$\Delta x = x_f - x_0 = 6.70 \text{ km} - 4.70 \text{ km} = +2.00 \text{ km}$$

3. Solve for displacement in part (b).

**Equation:**

$$\Delta x' = x'_f - x'_0 = 3.75 \text{ km} - 5.25 \text{ km} = -1.50 \text{ km}$$

**Discussion**

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a negative sign.

**Example:****Comparing Distance Traveled with Displacement: A Subway Train**

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in [\[link\]](#)?

**Strategy**

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in [\[link\]](#). Distance traveled is the total length of the path traveled between the two positions. (See [Displacement](#).) In the case of the subway train shown in [\[link\]](#), the distance traveled is the same as the distance between the initial and final positions of the train.

**Solution**

1. The displacement for part (a) was  $+2.00$  km. Therefore, the distance between the initial and final positions was  $2.00$  km, and the distance traveled was  $2.00$  km.
2. The displacement for part (b) was  $-1.5$  km. Therefore, the distance between the initial and final positions was  $1.50$  km, and the distance traveled was  $1.50$  km.

**Discussion**

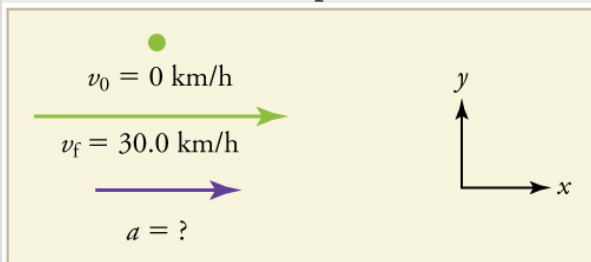
Distance is a scalar. It has magnitude but no sign to indicate direction.

**Example:****Calculating Acceleration: A Subway Train Speeding Up**

Suppose the train in [\[link\]](#)(a) accelerates from rest to  $30.0$  km/h in the first  $20.0$  s of its motion. What is its average acceleration during that time interval?

**Strategy**

It is worth it at this point to make a simple sketch:



This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

**Solution**

1. Identify the knowns.  $v_0 = 0$  (the train starts at rest),  $v_f = 30.0 \text{ km/h}$ , and  $\Delta t = 20.0 \text{ s}$ .
2. Calculate  $\Delta v$ . Since the train starts from rest, its change in velocity is  $\Delta v = +30.0 \text{ km/h}$ , where the plus sign means velocity to the right.
3. Plug in known values and solve for the unknown,  $\bar{a}$ .

**Equation:**

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+30.0 \text{ km/h}}{20.0 \text{ s}}$$

4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds. (See [Physical Quantities and Units](#) for more guidance.)

**Equation:**

$$\bar{a} = \left( \frac{+30 \text{ km/h}}{20.0 \text{ s}} \right) \left( \frac{10^3 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = 0.417 \text{ m/s}^2$$

**Discussion**

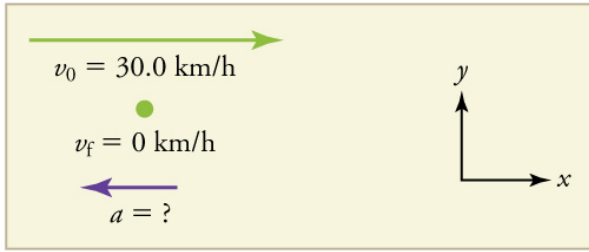
The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the *change* in velocity, as is always the case.

**Example:**

**Calculate Acceleration: A Subway Train Slowing Down**

Now suppose that at the end of its trip, the train in [\[link\]](#)(a) slows to a stop from a speed of  $30.0 \text{ km/h}$  in  $8.00 \text{ s}$ . What is its average acceleration while stopping?

**Strategy**



In this case, the train is decelerating and its acceleration is negative because it is toward the left. As in the previous example, we must find the change in velocity and the change in time and then solve for acceleration.

### Solution

1. Identify the knowns.  $v_0 = 30.0 \text{ km/h}$ ,  $v_f = 0 \text{ km/h}$  (the train is stopped, so its velocity is 0), and  $\Delta t = 8.00 \text{ s}$ .
2. Solve for the change in velocity,  $\Delta v$ .

### Equation:

$$\Delta v = v_f - v_0 = 0 - 30.0 \text{ km/h} = -30.0 \text{ km/h}$$

3. Plug in the knowns,  $\Delta v$  and  $\Delta t$ , and solve for  $\bar{a}$ .

### Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-30.0 \text{ km/h}}{8.00 \text{ s}}$$

4. Convert the units to meters and seconds.

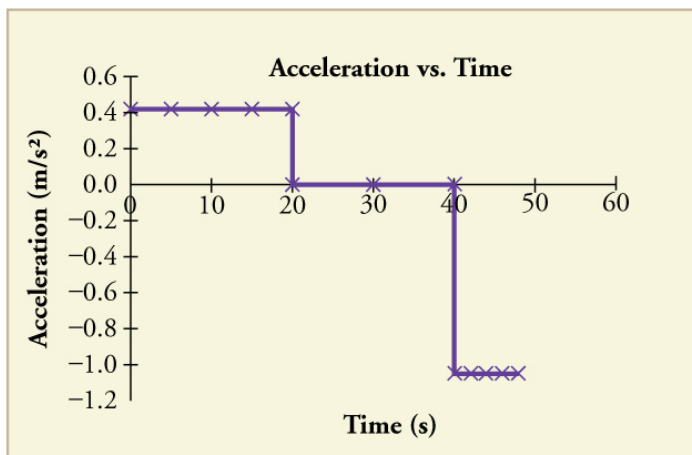
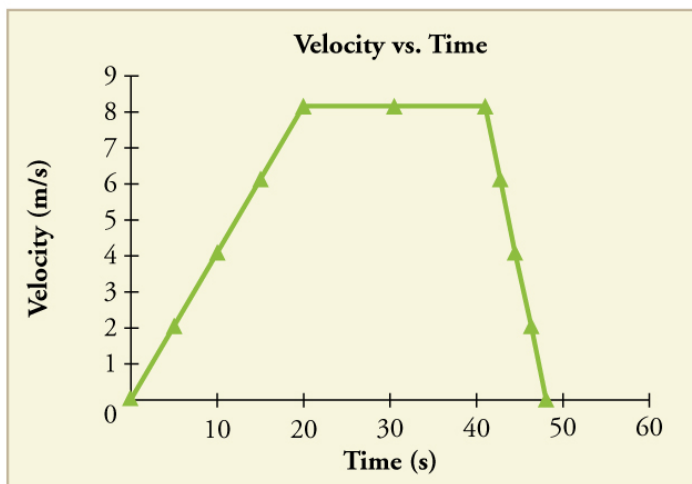
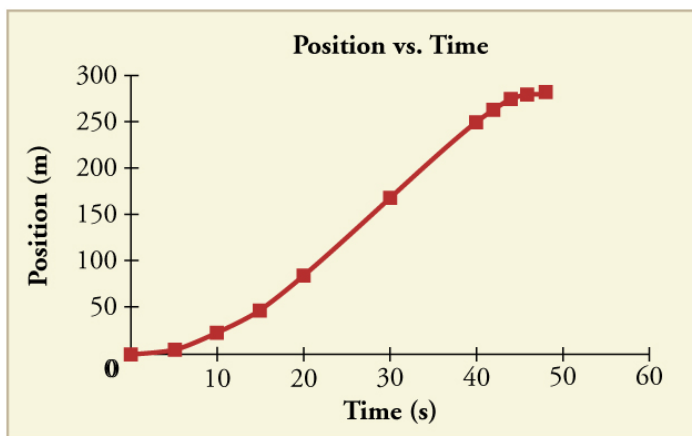
### Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \left( \frac{-30.0 \text{ km/h}}{8.00 \text{ s}} \right) \left( \frac{10^3 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = -1.04 \text{ m/s}^2.$$

### Discussion

The minus sign indicates that acceleration is to the left. This sign is reasonable because the train initially has a positive velocity in this problem, and a negative acceleration would oppose the motion. Again, acceleration is in the same direction as the *change* in velocity, which is negative here. This acceleration can be called a deceleration because it has a direction opposite to the velocity.

The graphs of position, velocity, and acceleration vs. time for the trains in [\[link\]](#) and [\[link\]](#) are displayed in [\[link\]](#). (We have taken the velocity to remain constant from 20 to 40 s, after which the train decelerates.)



(a) Position of the train over time.

Notice that the train's position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then changes more slowly as it slows down at the end of the journey. In the middle of the journey, while the velocity remains constant, the position changes at a constant rate. (b) Velocity

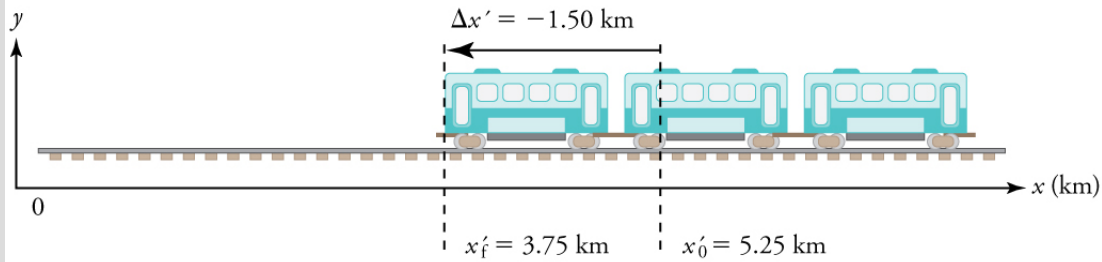
of the train over time. The train's velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the train decelerates at the end of the journey.

(c) The acceleration of the train over time. The train has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.

### **Example:**

#### **Calculating Average Velocity: The Subway Train**

What is the average velocity of the train in part b of [\[link\]](#), and shown again below, if it takes 5.00 min to make its trip?



### Strategy

Average velocity is displacement divided by time. It will be negative here, since the train moves to the left and has a negative displacement.

### Solution

1. Identify the knowns.  $x'_f = 3.75$  km,  $x'_0 = 5.25$  km,  $\Delta t = 5.00$  min.
2. Determine displacement,  $\Delta x'$ . We found  $\Delta x'$  to be  $-1.5$  km in [\[link\]](#).
3. Solve for average velocity.

### Equation:

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.50 \text{ km}}{5.00 \text{ min}}$$

4. Convert units.

### Equation:

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \left( \frac{-1.50 \text{ km}}{5.00 \text{ min}} \right) \left( \frac{60 \text{ min}}{1 \text{ h}} \right) = -18.0 \text{ km/h}$$

### Discussion

The negative velocity indicates motion to the left.

### Example:

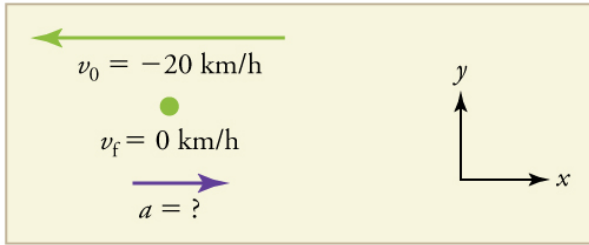
#### Calculating Deceleration: The Subway Train

Finally, suppose the train in [\[link\]](#) slows to a stop from a velocity of 20.0 km/h in 10.0 s. What is its average acceleration?

### Strategy

Once again, let's draw a sketch:





As before, we must find the change in velocity and the change in time to calculate average acceleration.

### Solution

1. Identify the knowns.  $v_0 = -20 \text{ km/h}$ ,  $v_f = 0 \text{ km/h}$ ,  $\Delta t = 10.0 \text{ s}$ .
2. Calculate  $\Delta v$ . The change in velocity here is actually positive, since

### Equation:

$$\Delta v = v_f - v_0 = 0 - (-20 \text{ km/h}) = +20 \text{ km/h}.$$

3. Solve for  $\bar{a}$ .

### Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+20.0 \text{ km/h}}{10.0 \text{ s}}$$

4. Convert units.

### Equation:

$$\bar{a} = \left( \frac{+20.0 \text{ km/h}}{10.0 \text{ s}} \right) \left( \frac{10^3 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = +0.556 \text{ m/s}^2$$

### Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train initially has a negative velocity (to the left) in this problem and a positive acceleration opposes the motion (and so it is to the right). Again, acceleration is in the same direction as the *change* in velocity, which is positive here. As in [\[link\]](#), this acceleration can be called a deceleration since it is in the direction opposite to the velocity.

## Sign and Direction

Perhaps the most important thing to note about these examples is the signs of the answers. In our chosen coordinate system, plus means the quantity is to the right and minus means it is to the left. This is easy to imagine for displacement and velocity. But it is a little less obvious for acceleration. Most people interpret negative acceleration as the slowing of an object. This was not the case in [\[link\]](#), where a positive acceleration slowed a negative velocity. The crucial distinction was that the acceleration was in the opposite direction from the velocity. In fact, a negative acceleration will *increase* a negative velocity. For example, the train moving to the left in [\[link\]](#) is sped up by an acceleration to the left. In that case, both  $v$  and  $a$  are negative. The plus and minus signs give the directions of the accelerations. If acceleration has the same sign as the velocity, the object is speeding up. If acceleration has the opposite sign as the velocity, the object is slowing down.

**Exercise:**

**Check Your Understanding**

**Problem:**

An airplane lands on a runway traveling east. Describe its acceleration.

---

**Solution:**

If we take east to be positive, then the airplane has negative acceleration, as it is accelerating toward the west. It is also decelerating: its acceleration is opposite in direction to its velocity.

**Note:**

**PhET Explorations: Moving Man Simulation**

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with the mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you.

<https://archive.cnx.org/specials/e2ca52af-8c6b-450e-ac2f-9300b38e8739/moving-man/>

## Section Summary

- Acceleration is the rate at which velocity changes. In symbols, **average acceleration**  $\bar{a}$  is

**Equation:**

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}.$$

- The SI unit for acceleration is  $\text{m/s}^2$ .
- Acceleration is a vector, and thus has both a magnitude and direction.
- Acceleration can be caused by either a change in the magnitude or the direction of the velocity.
- Instantaneous acceleration  $a$  is the acceleration at a specific instant in time.
- Deceleration is an acceleration with a direction opposite to that of the velocity.

## Conceptual Questions

**Exercise:**

**Problem:**

Is it possible for speed to be constant while acceleration is not zero?  
Give an example of such a situation.

**Exercise:**

**Problem:**

Is it possible for velocity to be constant while acceleration is not zero?  
Explain.

**Exercise:**

**Problem:**

Give an example in which velocity is zero yet acceleration is not.

**Exercise:****Problem:**

If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

**Exercise:****Problem:**

Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

**Problems & Exercises****Exercise:****Problem:**

A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?

---

**Solution:**

$$4.29 \text{ m/s}^2$$

**Exercise:****Problem: Professional Application**

Dr. John Paul Stapp was U.S. Air Force officer who studied the effects of extreme deceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s, and was brought jarringly back to rest in only 1.40 s! Calculate his (a) acceleration and (b) deceleration.

Express each in multiples of  $g$  ( $9.80 \text{ m/s}^2$ ) by taking its ratio to the acceleration of gravity.

**Exercise:**

**Problem:**

A commuter backs her car out of her garage with an acceleration of  $1.40 \text{ m/s}^2$ . (a) How long does it take her to reach a speed of  $2.00 \text{ m/s}$ ? (b) If she then brakes to a stop in  $0.800 \text{ s}$ , what is her deceleration?

---

**Solution:**

(a)  $1.43 \text{ s}$

(b)  $-2.50 \text{ m/s}^2$

**Exercise:**

**Problem:**

Assume that an intercontinental ballistic missile goes from rest to a suborbital speed of  $6.50 \text{ km/s}$  in  $60.0 \text{ s}$  (the actual speed and time are classified). What is its average acceleration in  $\text{m/s}^2$  and in multiples of  $g$  ( $9.80 \text{ m/s}^2$ )?

## Glossary

acceleration

the rate of change in velocity; the change in velocity over time

average acceleration

the change in velocity divided by the time over which it changes

instantaneous acceleration

acceleration at a specific point in time

deceleration

acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity

## Motion Equations for Constant Acceleration in One Dimension

- Calculate displacement of an object that is not accelerating, given initial position and velocity.
- Calculate final velocity of an accelerating object, given initial velocity, acceleration, and time.
- Calculate displacement and final position of an accelerating object, given initial position, initial velocity, time, and acceleration.



Kinematic equations can help us describe and predict the motion of moving objects such as these kayaks racing in Newbury, England. (credit: Barry Skeates, Flickr)

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration already covered.

**Notation:  $t$ ,  $x$ ,  $v$ ,  $a$**

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is  $\Delta t = t_f - t_0$ , taking  $t_0 = 0$  means that  $\Delta t = t_f$ , the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is,  $x_0$  is *the initial position* and  $v_0$  is *the initial velocity*. We put no subscripts on the final values. That is,  $t$  is *the final time*,  $x$  is *the final position*, and  $v$  is *the final velocity*. This gives a simpler expression for elapsed time—now,  $\Delta t = t$ . It also simplifies the expression for displacement, which is now  $\Delta x = x - x_0$ . Also, it simplifies the expression for change in velocity, which is now  $\Delta v = v - v_0$ . To summarize, using the simplified notation, with the initial time taken to be zero,

**Equation:**

$$\begin{aligned}\Delta t &= t \\ \Delta x &= x - x_0 \\ \Delta v &= v - v_0\end{aligned}$$

where *the subscript 0 denotes an initial value and the absence of a subscript denotes a final value* in whatever motion is under consideration.

We now make the important assumption that *acceleration is constant*. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

**Equation:**

$$\bar{a} = a = \text{constant},$$

so we use the symbol  $a$  for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the accuracy of our treatment. For one thing, acceleration is constant in a great number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in



motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

**Note:**

Solving for Displacement ( $\Delta x$ ) and Final Position ( $x$ ) from Average Velocity when Acceleration ( $a$ ) is Constant

To get our first two new equations, we start with the definition of average velocity:

**Equation:**

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

Substituting the simplified notation for  $\Delta x$  and  $\Delta t$  yields

**Equation:**

$$\bar{v} = \frac{x - x_0}{t}.$$

Solving for  $x$  yields

**Equation:**

$$x = x_0 + \bar{v}t,$$

where the average velocity is

**Equation:**

$$\bar{v} = \frac{v_0 + v}{2} \text{ (constant } a\text{)}.$$

The equation  $\bar{v} = \frac{v_0 + v}{2}$  reflects the fact that, when acceleration is constant,  $\bar{v}$  is just the simple average of the initial and final velocities. For example, if

you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation  $\bar{v} = \frac{v_0 + v}{2}$  to check this, we see that

**Equation:**

$$\bar{v} = \frac{v_0 + v}{2} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h},$$

which seems logical.

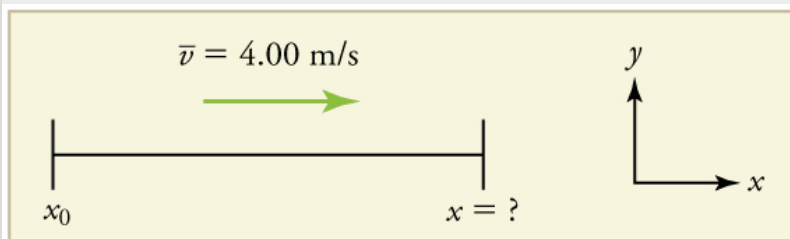
**Example:**

**Calculating Displacement: How Far does the Jogger Run?**

A jogger runs down a straight stretch of road with an average velocity of 4.00 m/s for 2.00 min. What is his final position, taking his initial position to be zero?

**Strategy**

Draw a sketch.



The final position  $x$  is given by the equation

**Equation:**

$$x = x_0 + \bar{v}t.$$

To find  $x$ , we identify the values of  $x_0$ ,  $\bar{v}$ , and  $t$  from the statement of the problem and substitute them into the equation.

**Solution**

1. Identify the knowns.  $\bar{v} = 4.00 \text{ m/s}$ ,  $\Delta t = 2.00 \text{ min}$ , and  $x_0 = 0 \text{ m}$ .
2. Enter the known values into the equation.

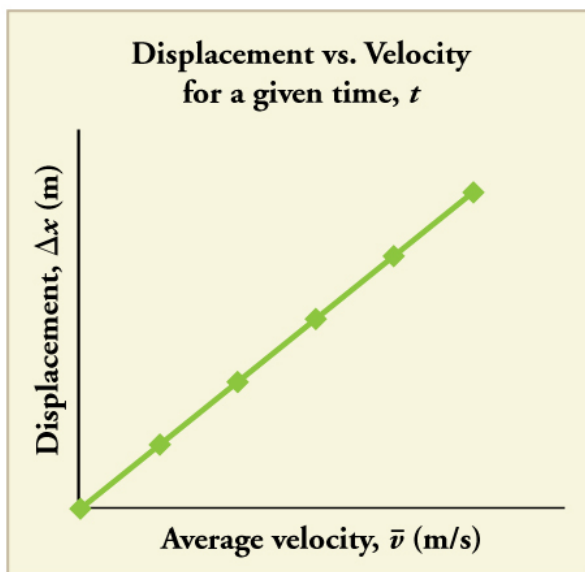
**Equation:**

$$x = x_0 + \bar{v}t = 0 + (4.00 \text{ m/s})(120 \text{ s}) = 480 \text{ m}$$

### Discussion

Velocity and final displacement are both positive, which means they are in the same direction.

The equation  $x = x_0 + \bar{v}t$  gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on  $\bar{v}$  rather than on  $\bar{v}$  raised to some other power, such as  $\bar{v}^2$ . When graphed, linear functions look like straight lines with a constant slope.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.



There is a linear relationship between displacement and average velocity. For a given time  $t$ , an object moving twice as fast as another object will

move twice as far as the other object.

**Note:**

**Solving for Final Velocity**

We can derive another useful equation by manipulating the definition of acceleration.

**Equation:**

$$a = \frac{\Delta v}{\Delta t}$$

Substituting the simplified notation for  $\Delta v$  and  $\Delta t$  gives us

**Equation:**

$$a = \frac{v - v_0}{t} \text{ (constant } a\text{)}.$$

Solving for  $v$  yields

**Equation:**

$$v = v_0 + at \text{ (constant } a\text{)}.$$

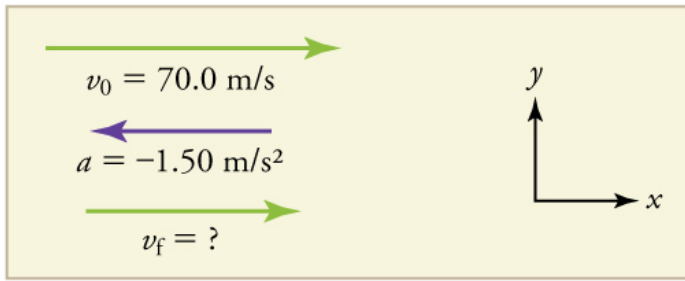
**Example:**

**Calculating Final Velocity: An Airplane Slowing Down after Landing**

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s<sup>2</sup> for 40.0 s. What is its final velocity?

**Strategy**

Draw a sketch. We draw the acceleration vector in the direction opposite the velocity vector because the plane is decelerating.



### Solution

1. Identify the knowns.  $v_0 = 70.0 \text{ m/s}$ ,  $a = -1.50 \text{ m/s}^2$ ,  $t = 40.0 \text{ s}$ .
2. Identify the unknown. In this case, it is final velocity,  $v_f$ .
3. Determine which equation to use. We can calculate the final velocity using the equation  $v = v_0 + at$ .
4. Plug in the known values and solve.

### Equation:

$$v = v_0 + at = 70.0 \text{ m/s} + (-1.50 \text{ m/s}^2)(40.0 \text{ s}) = 10.0 \text{ m/s}$$

### Discussion

The final velocity is much less than the initial velocity, as desired when slowing down, but still positive. With jet engines, reverse thrust could be maintained long enough to stop the plane and start moving it backward. That would be indicated by a negative final velocity, which is not the case here.



The airplane lands with an initial velocity of  $70.0 \text{ m/s}$  and slows to a final velocity of  $10.0 \text{ m/s}$  before heading for the terminal. Note that the acceleration is negative because its direction is opposite to its velocity, which is positive.

In addition to being useful in problem solving, the equation  $v = v_0 + at$  gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example, that

- final velocity depends on how large the acceleration is and how long it lasts
- if the acceleration is zero, then the final velocity equals the initial velocity ( $v = v_0$ ), as expected (i.e., velocity is constant)
- if  $a$  is negative, then the final velocity is less than the initial velocity

(All of these observations fit our intuition, and it is always useful to examine basic equations in light of our intuition and experiences to check that they do indeed describe nature accurately.)

**Note:**

Making Connections: Real-World Connection



The Space Shuttle *Endeavor* blasts off from the Kennedy Space Center in February 2010.  
(credit: Matthew Simantov, Flickr)

An intercontinental ballistic missile (ICBM) has a larger average acceleration than the Space Shuttle and achieves a greater velocity in the

first minute or two of flight (actual ICBM burn times are classified—short-burn-time missiles are more difficult for an enemy to destroy). But the Space Shuttle obtains a greater final velocity, so that it can orbit the earth rather than come directly back down as an ICBM does. The Space Shuttle does this by accelerating for a longer time.

**Note:**

Solving for Final Position When Velocity is Not Constant ( $a \neq 0$ )

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

**Equation:**

$$v = v_0 + at.$$

Adding  $v_0$  to each side of this equation and dividing by 2 gives

**Equation:**

$$\frac{v_0 + v}{2} = v_0 + \frac{1}{2}at.$$

Since  $\frac{v_0 + v}{2} = \bar{v}$  for constant acceleration, then

**Equation:**

$$\bar{v} = v_0 + \frac{1}{2}at.$$

Now we substitute this expression for  $\bar{v}$  into the equation for displacement,

$x = x_0 + \bar{v}t$ , yielding

**Equation:**

$$x = x_0 + v_0t + \frac{1}{2}at^2 \text{ (constant } a\text{)}.$$

**Example:****Calculating Displacement of an Accelerating Object: Dragsters**

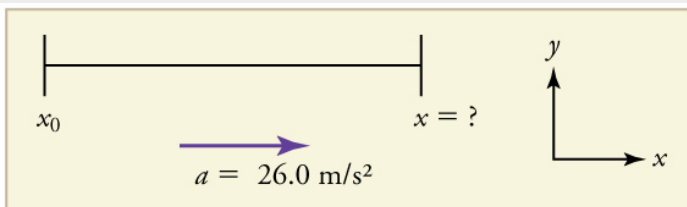
Dragsters can achieve average accelerations of  $26.0 \text{ m/s}^2$ . Suppose such a dragster accelerates from rest at this rate for  $5.56 \text{ s}$ . How far does it travel in this time?



U.S. Army Top Fuel pilot  
Tony “The Sarge”  
Schumacher begins a race  
with a controlled burnout.  
(credit: Lt. Col. William  
Thurmond. Photo  
Courtesy of U.S. Army.)

**Strategy**

Draw a sketch.



We are asked to find displacement, which is  $x$  if we take  $x_0$  to be zero. (Think about it like the starting line of a race. It can be anywhere, but we call it 0 and measure all other positions relative to it.) We can use the equation  $x = x_0 + v_0t + \frac{1}{2}at^2$  once we identify  $v_0$ ,  $a$ , and  $t$  from the statement of the problem.

**Solution**



1. Identify the knowns. Starting from rest means that  $v_0 = 0$ ,  $a$  is given as  $26.0 \text{ m/s}^2$  and  $t$  is given as  $5.56 \text{ s}$ .
2. Plug the known values into the equation to solve for the unknown  $x$ :

**Equation:**

$$x = x_0 + v_0 t + \frac{1}{2} a t^2.$$

Since the initial position and velocity are both zero, this simplifies to

**Equation:**

$$x = \frac{1}{2} a t^2.$$

Substituting the identified values of  $a$  and  $t$  gives

**Equation:**

$$x = \frac{1}{2} (26.0 \text{ m/s}^2) (5.56 \text{ s})^2,$$

yielding

**Equation:**

$$x = 402 \text{ m}.$$

### Discussion

If we convert  $402 \text{ m}$  to miles, we find that the distance covered is very close to one quarter of a mile, the standard distance for drag racing. So the answer is reasonable. This is an impressive displacement in only  $5.56 \text{ s}$ , but top-notch dragsters can do a quarter mile in even less time than this.

What else can we learn by examining the equation  $x = x_0 + v_0 t + \frac{1}{2} a t^2$ ?  
We see that:

- displacement depends on the square of the elapsed time when acceleration is not zero. In [\[link\]](#), the dragster covers only one fourth of the total distance in the first half of the elapsed time

- if acceleration is zero, then the initial velocity equals average velocity ( $v_0 = \bar{v}$ ) and  $x = x_0 + v_0 t + \frac{1}{2}at^2$  becomes  $x = x_0 + v_0 t$

**Note:**

Solving for Final Velocity when Velocity Is Not Constant ( $a \neq 0$ )

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve  $v = v_0 + at$  for  $t$ , we get

**Equation:**

$$t = \frac{v - v_0}{a}.$$

Substituting this and  $\bar{v} = \frac{v_0 + v}{2}$  into  $x = x_0 + \bar{v}t$ , we get

**Equation:**

$$v^2 = v_0^2 + 2a(x - x_0) \text{ (constant } a\text{)}.$$

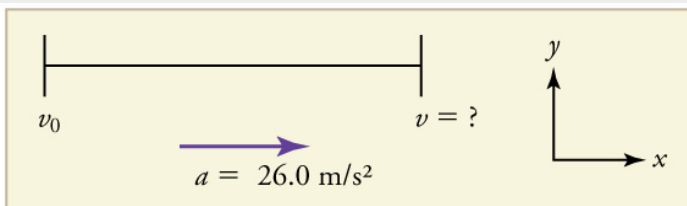
**Example:**

**Calculating Final Velocity: Dragsters**

Calculate the final velocity of the dragster in [\[link\]](#) without using information about time.

**Strategy**

Draw a sketch.



The equation  $v^2 = v_0^2 + 2a(x - x_0)$  is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

**Solution**

1. Identify the known values. We know that  $v_0 = 0$ , since the dragster starts from rest. Then we note that  $x - x_0 = 402$  m (this was the answer in [\[link\]](#)). Finally, the average acceleration was given to be  $a = 26.0$  m/s<sup>2</sup>.
2. Plug the knowns into the equation  $v^2 = v_0^2 + 2a(x - x_0)$  and solve for  $v$ .

**Equation:**

$$v^2 = 0 + 2(26.0 \text{ m/s}^2)(402 \text{ m}).$$

Thus

**Equation:**

$$v^2 = 2.09 \times 10^4 \text{ m}^2/\text{s}^2.$$

To get  $v$ , we take the square root:

**Equation:**

$$v = \sqrt{2.09 \times 10^4 \text{ m}^2/\text{s}^2} = 145 \text{ m/s}.$$

**Discussion**

145 m/s is about 522 km/h or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation  $v^2 = v_0^2 + 2a(x - x_0)$  can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts
- For a fixed deceleration, a car that is going twice as fast doesn't simply stop in twice the distance—it takes much further to stop. (This is why

we have reduced speed zones near schools.)

## Putting Equations Together

In the following examples, we further explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The box below provides easy reference to the equations needed.

**Note:**

Summary of Kinematic Equations (constant  $a$ )

**Equation:**

$$x = x_0 + \bar{v}t$$

**Equation:**

$$\bar{v} = \frac{v_0 + v}{2}$$

**Equation:**

$$v = v_0 + at$$

**Equation:**

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

**Equation:**

$$v^2 = v_0^2 + 2a(x - x_0)$$

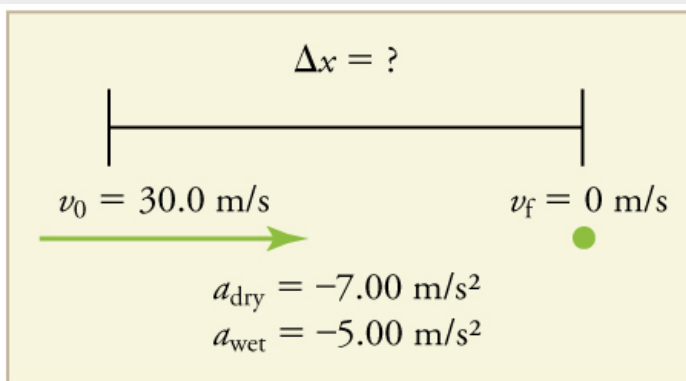
**Example:**

## Calculating Displacement: How Far Does a Car Go When Coming to a Halt?

On dry concrete, a car can decelerate at a rate of  $7.00 \text{ m/s}^2$ , whereas on wet concrete it can decelerate at only  $5.00 \text{ m/s}^2$ . Find the distances necessary to stop a car moving at  $30.0 \text{ m/s}$  (about  $110 \text{ km/h}$ ) (a) on dry concrete and (b) on wet concrete. (c) Repeat both calculations, finding the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of  $0.500 \text{ s}$  to get his foot on the brake.

### Strategy

Draw a sketch.



In order to determine which equations are best to use, we need to list all of the known values and identify exactly what we need to solve for. We shall do this explicitly in the next several examples, using tables to set them off.

### Solution for (a)

1. Identify the knowns and what we want to solve for. We know that  $v_0 = 30.0 \text{ m/s}$ ;  $v = 0$ ;  $a = -7.00 \text{ m/s}^2$  ( $a$  is negative because it is in a direction opposite to velocity). We take  $x_0$  to be 0. We are looking for displacement  $\Delta x$ , or  $x - x_0$ .

2. Identify the equation that will help up solve the problem. The best equation to use is

### Equation:

$$v^2 = v_0^2 + 2a(x - x_0).$$

This equation is best because it includes only one unknown,  $x$ . We know the values of all the other variables in this equation. (There are other equations that would allow us to solve for  $x$ , but they require us to know

the stopping time,  $t$ , which we do not know. We could use them but it would entail additional calculations.)

3. Rearrange the equation to solve for  $x$ .

**Equation:**

$$x - x_0 = \frac{v^2 - v_0^2}{2a}$$

4. Enter known values.

**Equation:**

$$x - 0 = \frac{0^2 - (30.0 \text{ m/s})^2}{2(-7.00 \text{ m/s}^2)}$$

Thus,

**Equation:**

$$x = 64.3 \text{ m on dry concrete.}$$

### **Solution for (b)**

This part can be solved in exactly the same manner as Part A. The only difference is that the deceleration is  $-5.00 \text{ m/s}^2$ . The result is

**Equation:**

$$x_{\text{wet}} = 90.0 \text{ m on wet concrete.}$$

### **Solution for (c)**

Once the driver reacts, the stopping distance is the same as it is in Parts A and B for dry and wet concrete. So to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume that the velocity remains constant during the driver's reaction time.

1. Identify the knowns and what we want to solve for. We know that

$\bar{v} = 30.0 \text{ m/s}$ ;  $t_{\text{reaction}} = 0.500 \text{ s}$ ;  $a_{\text{reaction}} = 0$ . We take  $x_{0-\text{reaction}}$  to be 0. We are looking for  $x_{\text{reaction}}$ .

2. Identify the best equation to use.

$x = x_0 + \bar{v}t$  works well because the only unknown value is  $x$ , which is what we want to solve for.

3. Plug in the knowns to solve the equation.

**Equation:**

$$x = 0 + (30.0 \text{ m/s})(0.500 \text{ s}) = 15.0 \text{ m}.$$

This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly.

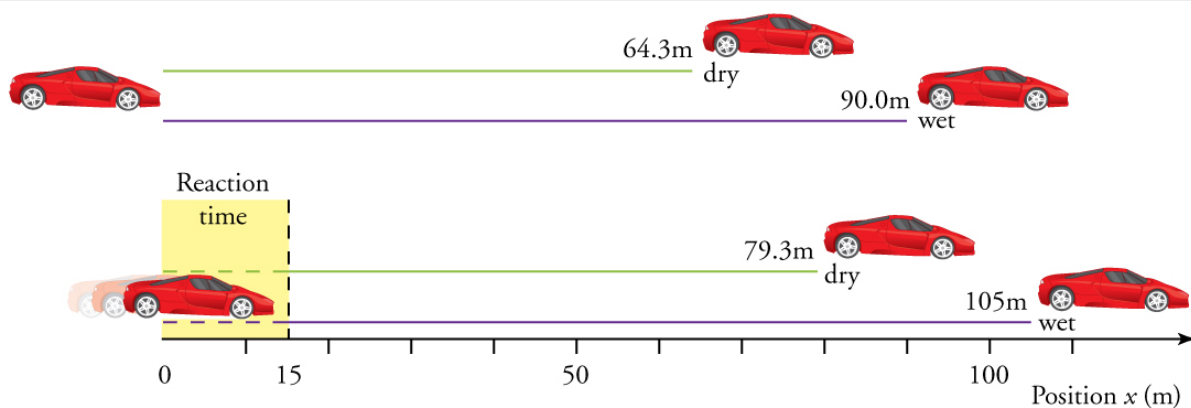
4. Add the displacement during the reaction time to the displacement when braking.

**Equation:**

$$x_{\text{braking}} + x_{\text{reaction}} = x_{\text{total}}$$

a.  $64.3 \text{ m} + 15.0 \text{ m} = 79.3 \text{ m}$  when dry

b.  $90.0 \text{ m} + 15.0 \text{ m} = 105 \text{ m}$  when wet



The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car initially traveling at 30.0 m/s. Also shown are the total distances traveled from the point where the driver first sees a light turn red, assuming a 0.500 s reaction time.

## Discussion

The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet rather than dry pavement. It is interesting that reaction time adds significantly to the displacements. But more important is the general approach to solving problems. We identify the knowns and the quantities to be determined and then find an appropriate equation. There is often more than one way to solve a problem. The various parts of this example can in fact be solved by other methods, but the solutions presented above are the shortest.

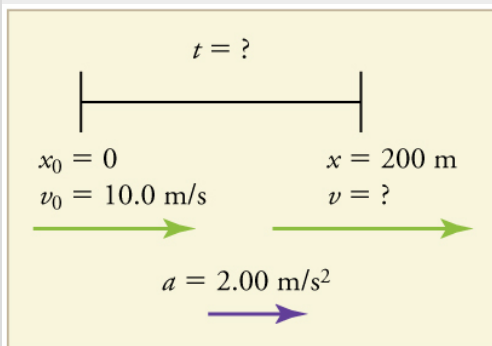
## Example:

### Calculating Time: A Car Merges into Traffic

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at  $2.00 \text{ m/s}^2$ , how long does it take to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

### Strategy

Draw a sketch.



We are asked to solve for the time  $t$ . As before, we identify the known quantities in order to choose a convenient physical relationship (that is, an equation with one unknown,  $t$ ).

### Solution

1. Identify the knowns and what we want to solve for. We know that  $v_0 = 10 \text{ m/s}$ ;  $a = 2.00 \text{ m/s}^2$ ; and  $x = 200 \text{ m}$ .
2. We need to solve for  $t$ . Choose the best equation.  $x = x_0 + v_0 t + \frac{1}{2} a t^2$  works best because the only unknown in the equation is the variable  $t$  for which we need to solve.



3. We will need to rearrange the equation to solve for  $t$ . In this case, it will be easier to plug in the knowns first.

**Equation:**

$$200 \text{ m} = 0 \text{ m} + (10.0 \text{ m/s})t + \frac{1}{2} (2.00 \text{ m/s}^2) t^2$$

4. Simplify the equation. The units of meters (m) cancel because they are in each term. We can get the units of seconds (s) to cancel by taking  $t = t \text{ s}$ , where  $t$  is the magnitude of time and s is the unit. Doing so leaves

**Equation:**

$$200 = 10t + t^2.$$

5. Use the quadratic formula to solve for  $t$ .

(a) Rearrange the equation to get 0 on one side of the equation.

**Equation:**

$$t^2 + 10t - 200 = 0$$

This is a quadratic equation of the form

**Equation:**

$$at^2 + bt + c = 0,$$

where the constants are  $a = 1.00$ ,  $b = 10.0$ , and  $c = -200$ .

(b) Its solutions are given by the quadratic formula:

**Equation:**

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This yields two solutions for  $t$ , which are

**Equation:**

$$t = 10.0 \text{ and } -20.0.$$

In this case, then, the time is  $t = t$  in seconds, or

**Equation:**

$$t = 10.0 \text{ s and } -20.0 \text{ s.}$$

A negative value for time is unreasonable, since it would mean that the event happened 20 s before the motion began. We can discard that solution. Thus,

**Equation:**

$$t = 10.0 \text{ s.}$$

### Discussion

Whenever an equation contains an unknown squared, there will be two solutions. In some problems both solutions are meaningful, but in others, such as the above, only one solution is reasonable. The 10.0 s answer seems reasonable for a typical freeway on-ramp.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. [Problem-Solving Basics](#) discusses problem-solving basics and outlines an approach that will help you succeed in this invaluable task.

### Note:

#### Making Connections: Take-Home Experiment—Breaking News

We have been using SI units of meters per second squared to describe some examples of acceleration or deceleration of cars, runners, and trains. To achieve a better feel for these numbers, one can measure the braking deceleration of a car doing a slow (and safe) stop. Recall that, for average acceleration,  $\bar{a} = \Delta v / \Delta t$ . While traveling in a car, slowly apply the brakes as you come up to a stop sign. Have a passenger note the initial speed in miles per hour and the time taken (in seconds) to stop. From this, calculate the deceleration in miles per hour per second. Convert this to meters per second squared and compare with other decelerations mentioned in this chapter. Calculate the distance traveled in braking.

**Exercise:**  
**Check Your Understanding**

**Problem:**

A manned rocket accelerates at a rate of  $20 \text{ m/s}^2$  during launch. How long does it take the rocket to reach a velocity of  $400 \text{ m/s}$ ?

---

**Solution:**

To answer this, choose an equation that allows you to solve for time  $t$ , given only  $a$ ,  $v_0$ , and  $v$ .

**Equation:**

$$v = v_0 + at$$

Rearrange to solve for  $t$ .

**Equation:**

$$t = \frac{v - v_0}{a} = \frac{400 \text{ m/s} - 0 \text{ m/s}}{20 \text{ m/s}^2} = 20 \text{ s}$$

## Section Summary

- To simplify calculations we take acceleration to be constant, so that  $\bar{a} = a$  at all times.
- We also take initial time to be zero.
- Initial position and velocity are given a subscript 0; final values have no subscript. Thus,

**Equation:**

$$\Delta t = t$$

$$\Delta x = x - x_0$$

$$\Delta v = v - v_0$$

- The following kinematic equations for motion with constant  $a$  are useful:

**Equation:**

$$x = x_0 + \bar{v}t$$

**Equation:**

$$\bar{v} = \frac{v_0 + v}{2}$$

**Equation:**

$$v = v_0 + at$$

**Equation:**

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

**Equation:**

$$v^2 = v_0^2 + 2a(x - x_0)$$

- In vertical motion,  $y$  is substituted for  $x$ .

## Problems & Exercises

**Exercise:**

**Problem:**

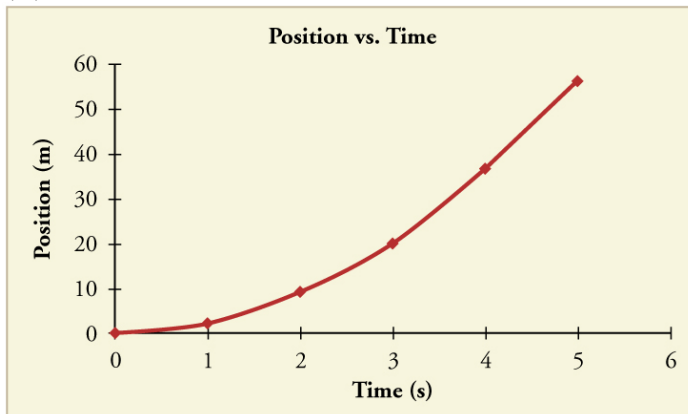
An Olympic-class sprinter starts a race with an acceleration of  $4.50 \text{ m/s}^2$ . (a) What is her speed 2.40 s later? (b) Sketch a graph of her position vs. time for this period.

---

**Solution:**

(a)  $10.8 \text{ m/s}$

(b)



**Exercise:**

**Problem:**

A well-thrown ball is caught in a well-padded mitt. If the deceleration of the ball is  $2.10 \times 10^4 \text{ m/s}^2$ , and 1.85 ms ( $1 \text{ ms} = 10^{-3} \text{ s}$ ) elapses from the time the ball first touches the mitt until it stops, what was the initial velocity of the ball?

---

**Solution:**

38.9 m/s (about 87 miles per hour)

**Exercise:**

**Problem:**

A bullet in a gun is accelerated from the firing chamber to the end of the barrel at an average rate of  $6.20 \times 10^5 \text{ m/s}^2$  for  $8.10 \times 10^{-4} \text{ s}$ . What is its muzzle velocity (that is, its final velocity)?

**Exercise:**

**Problem:**

(a) A light-rail commuter train accelerates at a rate of  $1.35 \text{ m/s}^2$ . How long does it take to reach its top speed of  $80.0 \text{ km/h}$ , starting from rest? (b) The same train ordinarily decelerates at a rate of  $1.65 \text{ m/s}^2$ . How long does it take to come to a stop from its top speed? (c) In emergencies the train can decelerate more rapidly, coming to rest from  $80.0 \text{ km/h}$  in  $8.30 \text{ s}$ . What is its emergency deceleration in  $\text{m/s}^2$ ?

---

**Solution:**

(a)  $16.5 \text{ s}$

(b)  $13.5 \text{ s}$

(c)  $-2.68 \text{ m/s}^2$

**Exercise:****Problem:**

While entering a freeway, a car accelerates from rest at a rate of  $2.40 \text{ m/s}^2$  for  $12.0 \text{ s}$ . (a) Draw a sketch of the situation. (b) List the knowns in this problem. (c) How far does the car travel in those  $12.0 \text{ s}$ ? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, check your units, and discuss whether the answer is reasonable. (d) What is the car's final velocity? Solve for this unknown in the same manner as in part (c), showing all steps explicitly.

**Exercise:****Problem:**

At the end of a race, a runner decelerates from a velocity of  $9.00 \text{ m/s}$  at a rate of  $2.00 \text{ m/s}^2$ . (a) How far does she travel in the next  $5.00 \text{ s}$ ? (b) What is her final velocity? (c) Evaluate the result. Does it make sense?

---

**Solution:**

(a) 20.0 m

(b)  $-1.00 \text{ m/s}$

(c) This result does not really make sense. If the runner starts at  $9.00 \text{ m/s}$  and decelerates at  $2.00 \text{ m/s}^2$ , then she will have stopped after  $4.50 \text{ s}$ . If she continues to decelerate, she will be running backwards.

**Exercise:****Problem: Professional Application:**

Blood is accelerated from rest to  $30.0 \text{ cm/s}$  in a distance of  $1.80 \text{ cm}$  by the left ventricle of the heart. (a) Make a sketch of the situation. (b) List the knowns in this problem. (c) How long does the acceleration take? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking your units. (d) Is the answer reasonable when compared with the time for a heartbeat?

**Exercise:****Problem:**

In a slap shot, a hockey player accelerates the puck from a velocity of  $8.00 \text{ m/s}$  to  $40.0 \text{ m/s}$  in the same direction. If this shot takes  $3.33 \times 10^{-2} \text{ s}$ , calculate the distance over which the puck accelerates.

---

**Solution:**

$0.799 \text{ m}$

**Exercise:**

**Problem:**

A powerful motorcycle can accelerate from rest to 26.8 m/s (100 km/h) in only 3.90 s. (a) What is its average acceleration? (b) How far does it travel in that time?

**Exercise:****Problem:**

Freight trains can produce only relatively small accelerations and decelerations. (a) What is the final velocity of a freight train that accelerates at a rate of  $0.0500 \text{ m/s}^2$  for 8.00 min, starting with an initial velocity of 4.00 m/s? (b) If the train can slow down at a rate of  $0.550 \text{ m/s}^2$ , how long will it take to come to a stop from this velocity? (c) How far will it travel in each case?

---

**Solution:**

(a) 28.0 m/s

(b) 50.9 s

(c) 7.68 km to accelerate and 713 m to decelerate

**Exercise:****Problem:**

A fireworks shell is accelerated from rest to a velocity of 65.0 m/s over a distance of 0.250 m. (a) How long did the acceleration last? (b) Calculate the acceleration.

**Exercise:**



**Problem:**

A swan on a lake gets airborne by flapping its wings and running on top of the water. (a) If the swan must reach a velocity of  $6.00 \text{ m/s}$  to take off and it accelerates from rest at an average rate of  $0.350 \text{ m/s}^2$ , how far will it travel before becoming airborne? (b) How long does this take?

---

**Solution:**

(a)  $51.4 \text{ m}$

(b)  $17.1 \text{ s}$

**Exercise:****Problem: Professional Application:**

A woodpecker's brain is specially protected from large decelerations by tendon-like attachments inside the skull. While pecking on a tree, the woodpecker's head comes to a stop from an initial velocity of  $0.600 \text{ m/s}$  in a distance of only  $2.00 \text{ mm}$ . (a) Find the acceleration in  $\text{m/s}^2$  and in multiples of  $g$  ( $g = 9.80 \text{ m/s}^2$ ). (b) Calculate the stopping time. (c) The tendons cradling the brain stretch, making its stopping distance  $4.50 \text{ mm}$  (greater than the head and, hence, less deceleration of the brain). What is the brain's deceleration, expressed in multiples of  $g$ ?

**Exercise:****Problem:**

An unwary football player collides with a padded goalpost while running at a velocity of  $7.50 \text{ m/s}$  and comes to a full stop after compressing the padding and his body  $0.350 \text{ m}$ . (a) What is his deceleration? (b) How long does the collision last?

---

**Solution:**

(a)  $-80.4 \text{ m/s}^2$

(b)  $9.33 \times 10^{-2} \text{ s}$

**Exercise:**

**Problem:**

In World War II, there were several reported cases of airmen who jumped from their flaming airplanes with no parachute to escape certain death. Some fell about 20,000 feet (6000 m), and some of them survived, with few life-threatening injuries. For these lucky pilots, the tree branches and snow drifts on the ground allowed their deceleration to be relatively small. If we assume that a pilot's speed upon impact was 123 mph (54 m/s), then what was his deceleration? Assume that the trees and snow stopped him over a distance of 3.0 m.

**Exercise:**

**Problem:**

Consider a grey squirrel falling out of a tree to the ground. (a) If we ignore air resistance in this case (only for the sake of this problem), determine a squirrel's velocity just before hitting the ground, assuming it fell from a height of 3.0 m. (b) If the squirrel stops in a distance of 2.0 cm through bending its limbs, compare its deceleration with that of the airman in the previous problem.

---

**Solution:**

(a)  $7.7 \text{ m/s}$

(b)  $-15 \times 10^2 \text{ m/s}^2$ . This is about 3 times the deceleration of the pilots, who were falling from thousands of meters high!

**Exercise:**

**Problem:**

An express train passes through a station. It enters with an initial velocity of  $22.0 \text{ m/s}$  and decelerates at a rate of  $0.150 \text{ m/s}^2$  as it goes through. The station is  $210 \text{ m}$  long. (a) How long is the nose of the train in the station? (b) How fast is it going when the nose leaves the station? (c) If the train is  $130 \text{ m}$  long, when does the end of the train leave the station? (d) What is the velocity of the end of the train as it leaves?

**Exercise:****Problem:**

Dragsters can actually reach a top speed of  $145 \text{ m/s}$  in only  $4.45 \text{ s}$ —considerably less time than given in [\[link\]](#) and [\[link\]](#). (a) Calculate the average acceleration for such a dragster. (b) Find the final velocity of this dragster starting from rest and accelerating at the rate found in (a) for  $402 \text{ m}$  (a quarter mile) without using any information on time. (c) Why is the final velocity greater than that used to find the average acceleration? *Hint:* Consider whether the assumption of constant acceleration is valid for a dragster. If not, discuss whether the acceleration would be greater at the beginning or end of the run and what effect that would have on the final velocity.

---

**Solution:**

(a)  $32.6 \text{ m/s}^2$

(b)  $162 \text{ m/s}$

(c)  $v > v_{\text{max}}$ , because the assumption of constant acceleration is not valid for a dragster. A dragster changes gears, and would have a greater acceleration in first gear than second gear than third gear, etc. The acceleration would be greatest at the beginning, so it would not be accelerating at  $32.6 \text{ m/s}^2$  during the last few meters, but substantially less, and the final velocity would be less than  $162 \text{ m/s}$ .

**Exercise:****Problem:**

A bicycle racer sprints at the end of a race to clinch a victory. The racer has an initial velocity of  $11.5 \text{ m/s}$  and accelerates at the rate of  $0.500 \text{ m/s}^2$  for  $7.00 \text{ s}$ . (a) What is his final velocity? (b) The racer continues at this velocity to the finish line. If he was  $300 \text{ m}$  from the finish line when he started to accelerate, how much time did he save? (c) One other racer was  $5.00 \text{ m}$  ahead when the winner started to accelerate, but he was unable to accelerate, and traveled at  $11.8 \text{ m/s}$  until the finish line. How far ahead of him (in meters and in seconds) did the winner finish?

**Exercise:****Problem:**

In 1967, New Zealander Burt Munro set the world record for an Indian motorcycle, on the Bonneville Salt Flats in Utah, with a maximum speed of  $183.58 \text{ mi/h}$ . The one-way course was  $5.00 \text{ mi}$  long. Acceleration rates are often described by the time it takes to reach  $60.0 \text{ mi/h}$  from rest. If this time was  $4.00 \text{ s}$ , and Burt accelerated at this rate until he reached his maximum speed, how long did it take Burt to complete the course?

---

**Solution:**

$104 \text{ s}$

**Exercise:**

**Problem:**

(a) A world record was set for the men's 100-m dash in the 2008 Olympic Games in Beijing by Usain Bolt of Jamaica. Bolt "coasted" across the finish line with a time of 9.69 s. If we assume that Bolt accelerated for 3.00 s to reach his maximum speed, and maintained that speed for the rest of the race, calculate his maximum speed and his acceleration. (b) During the same Olympics, Bolt also set the world record in the 200-m dash with a time of 19.30 s. Using the same assumptions as for the 100-m dash, what was his maximum speed for this race?

---

**Solution:**

(a)  $v = 12.2 \text{ m/s}$ ;  $a = 4.07 \text{ m/s}^2$

(b)  $v = 11.2 \text{ m/s}$

## Problem-Solving Basics for One-Dimensional Kinematics

- Apply problem-solving steps and strategies to solve problems of one-dimensional kinematics.
- Apply strategies to determine whether or not the result of a problem is reasonable, and if not, determine the cause.



Problem-solving skills are essential to your success in Physics. (credit: scui3asteveo, Flickr)

Problem-solving skills are obviously essential to success in a quantitative course in physics. More importantly, the ability to apply broad physical principles, usually represented by equations, to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations, whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in this text and for applying physics in everyday and professional life.

## Problem-Solving Steps

While there is no simple step-by-step method that works for every problem, the following general procedures facilitate problem solving and make it more meaningful. A certain amount of creativity and insight is required as well.

### **Step 1**

*Examine the situation to determine which physical principles are involved.* It often helps to *draw a simple sketch* at the outset. You will also need to decide which direction is positive and note that on your sketch. Once you have identified the physical principles, it is much easier to find and apply the equations representing those principles. Although finding the correct equation is essential, keep in mind that equations represent physical principles, laws of nature, and relationships among physical quantities. Without a conceptual understanding of a problem, a numerical solution is meaningless.

### **Step 2**

*Make a list of what is given or can be inferred from the problem as stated (identify the knowns).* Many problems are stated very succinctly and require some inspection to determine what is known. A sketch can also be very useful at this point. Formally identifying the knowns is of particular importance in applying physics to real-world situations. Remember, “stopped” means velocity is zero, and we often can take initial time and position as zero.

### **Step 3**

*Identify exactly what needs to be determined in the problem (identify the unknowns).* In complex problems, especially, it is not always obvious what needs to be found or in what sequence. Making a list can help.

## Step 4

*Find an equation or set of equations that can help you solve the problem.*

Your list of knowns and unknowns can help here. It is easiest if you can find equations that contain only one unknown—that is, all of the other variables are known, so you can easily solve for the unknown. If the equation contains more than one unknown, then an additional equation is needed to solve the problem. In some problems, several unknowns must be determined to get at the one needed most. In such problems it is especially important to keep physical principles in mind to avoid going astray in a sea of equations. You may have to use two (or more) different equations to get the final answer.

## Step 5

*Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.* This step produces the numerical answer; it also provides a check on units that can help you find errors. If the units of the answer are incorrect, then an error has been made. However, be warned that correct units do not guarantee that the numerical part of the answer is also correct.

## Step 6

*Check the answer to see if it is reasonable: Does it make sense?* This final step is extremely important—the goal of physics is to accurately describe nature. To see if the answer is reasonable, check both its magnitude and its sign, in addition to its units. Your judgment will improve as you solve more and more physics problems, and it will become possible for you to make finer and finer judgments regarding whether nature is adequately described by the answer to a problem. This step brings the problem back to its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than just being able to mechanically solve a problem.



When solving problems, we often perform these steps in different order, and we also tend to do several steps simultaneously. There is no rigid procedure that will work every time. Creativity and insight grow with experience, and the basics of problem solving become almost automatic. One way to get practice is to work out the text's examples for yourself as you read. Another is to work as many end-of-section problems as possible, starting with the easiest to build confidence and progressing to the more difficult. Once you become involved in physics, you will see it all around you, and you can begin to apply it to situations you encounter outside the classroom, just as is done in many of the applications in this text.

## Unreasonable Results

Physics must describe nature accurately. Some problems have results that are unreasonable because one premise is unreasonable or because certain premises are inconsistent with one another. The physical principle applied correctly then produces an unreasonable result. For example, if a person starting a foot race accelerates at  $0.40 \text{ m/s}^2$  for 100 s, his final speed will be 40 m/s (about 150 km/h)—clearly unreasonable because the time of 100 s is an unreasonable premise. The physics is correct in a sense, but there is more to describing nature than just manipulating equations correctly. Checking the result of a problem to see if it is reasonable does more than help uncover errors in problem solving—it also builds intuition in judging whether nature is being accurately described.

Use the following strategies to determine whether an answer is reasonable and, if it is not, to determine what is the cause.

### Step 1

*Solve the problem using strategies as outlined and in the format followed in the worked examples in the text.* In the example given in the preceding paragraph, you would identify the givens as the acceleration and time and use the equation below to find the unknown final velocity. That is,

**Equation:**

$$v = v_0 + at = 0 + (0.40 \text{ m/s}^2)(100 \text{ s}) = 40 \text{ m/s}.$$

## Step 2

*Check to see if the answer is reasonable.* Is it too large or too small, or does it have the wrong sign, improper units, ...? In this case, you may need to convert meters per second into a more familiar unit, such as miles per hour.

**Equation:**

$$\left(\frac{40 \text{ m}}{\text{s}}\right) \left(\frac{3.28 \text{ ft}}{\text{m}}\right) \left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right) \left(\frac{60 \text{ s}}{\text{min}}\right) \left(\frac{60 \text{ min}}{1 \text{ h}}\right) = 89 \text{ mph}$$

This velocity is about four times greater than a person can run—so it is too large.

## Step 3

*If the answer is unreasonable, look for what specifically could cause the identified difficulty.* In the example of the runner, there are only two assumptions that are suspect. The acceleration could be too great or the time too long. First look at the acceleration and think about what the number means. If someone accelerates at  $0.40 \text{ m/s}^2$ , their velocity is increasing by  $0.4 \text{ m/s}$  each second. Does this seem reasonable? If so, the time must be too long. It is not possible for someone to accelerate at a constant rate of  $0.40 \text{ m/s}^2$  for  $100 \text{ s}$  (almost two minutes).

## Section Summary

- *The six basic problem solving steps for physics are:*

*Step 1.* Examine the situation to determine which physical principles are involved.

*Step 2.* Make a list of what is given or can be inferred from the problem as stated (identify the knowns).

*Step 3.* Identify exactly what needs to be determined in the problem (identify the unknowns).

*Step 4.* Find an equation or set of equations that can help you solve the problem.

*Step 5.* Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.

*Step 6.* Check the answer to see if it is reasonable: Does it make sense?

## Conceptual Questions

### Exercise:

#### Problem:

What information do you need in order to choose which equation or equations to use to solve a problem? Explain.

### Exercise:

#### Problem:

What is the last thing you should do when solving a problem? Explain.

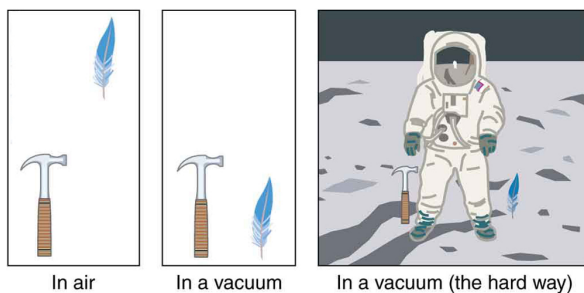
## Falling Objects

- Describe the effects of gravity on objects in motion.
- Describe the motion of objects that are in free fall.
- Calculate the position and velocity of objects in free fall.

Falling objects form an interesting class of motion problems. For example, we can estimate the depth of a vertical mine shaft by dropping a rock into it and listening for the rock to hit the bottom. By applying the kinematics developed so far to falling objects, we can examine some interesting situations and learn much about gravity in the process.

## Gravity

The most remarkable and unexpected fact about falling objects is that, if air resistance and friction are negligible, then in a given location all objects fall toward the center of Earth with the *same constant acceleration, independent of their mass*. This experimentally determined fact is unexpected, because we are so accustomed to the effects of air resistance and friction that we expect light objects to fall slower than heavy ones.



A hammer and a feather will fall with the same constant acceleration if air resistance is considered negligible. This is a general characteristic of gravity not unique to Earth, as astronaut David R. Scott demonstrated on the Moon in 1971, where the

acceleration due to gravity is  
only  $1.67 \text{ m/s}^2$ .

In the real world, air resistance can cause a lighter object to fall slower than a heavier object of the same size. A tennis ball will reach the ground after a hard baseball dropped at the same time. (It might be difficult to observe the difference if the height is not large.) Air resistance opposes the motion of an object through the air, while friction between objects—such as between clothes and a laundry chute or between a stone and a pool into which it is dropped—also opposes motion between them. For the ideal situations of these first few chapters, an object *falling without air resistance or friction* is defined to be in **free-fall**.

The force of gravity causes objects to fall toward the center of Earth. The acceleration of free-falling objects is therefore called the **acceleration due to gravity**. The acceleration due to gravity is *constant*, which means we can apply the kinematics equations to any falling object where air resistance and friction are negligible. This opens a broad class of interesting situations to us. The acceleration due to gravity is so important that its magnitude is given its own symbol,  $g$ . It is constant at any given location on Earth and has the average value

**Equation:**

$$g = 9.80 \text{ m/s}^2.$$

Although  $g$  varies from  $9.78 \text{ m/s}^2$  to  $9.83 \text{ m/s}^2$ , depending on latitude, altitude, underlying geological formations, and local topography, the average value of  $9.80 \text{ m/s}^2$  will be used in this text unless otherwise specified. The direction of the acceleration due to gravity is *downward (towards the center of Earth)*. In fact, its direction *defines* what we call vertical. Note that whether the acceleration  $a$  in the kinematic equations has the value  $+g$  or  $-g$  depends on how we define our coordinate system. If we define the upward direction as positive, then  $a = -g = -9.80 \text{ m/s}^2$ , and if we define the downward direction as positive, then  $a = g = 9.80 \text{ m/s}^2$ .

## One-Dimensional Motion Involving Gravity

The best way to see the basic features of motion involving gravity is to start with the simplest situations and then progress toward more complex ones. So we start by considering straight up and down motion with no air resistance or friction. These assumptions mean that the velocity (if there is any) is vertical. If the object is dropped, we know the initial velocity is zero. Once the object has left contact with whatever held or threw it, the object is in free-fall. Under these circumstances, the motion is one-dimensional and has constant acceleration of magnitude  $g$ . We will also represent vertical displacement with the symbol  $y$  and use  $x$  for horizontal displacement.

### **Note:**

Kinematic Equations for Objects in Free-Fall where Acceleration =  $-g$

### **Equation:**

$$v = v_0 - gt$$

### **Equation:**

$$y = y_0 + v_0t - \frac{1}{2}gt^2$$

### **Equation:**

$$v^2 = v_0^2 - 2g(y - y_0)$$

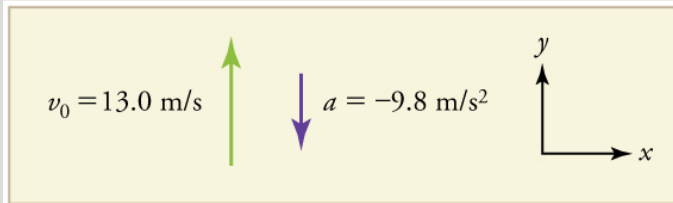
### **Example:**

#### **Calculating Position and Velocity of a Falling Object: A Rock Thrown Upward**

A person standing on the edge of a high cliff throws a rock straight up with an initial velocity of 13.0 m/s. The rock misses the edge of the cliff as it falls back to earth. Calculate the position and velocity of the rock 1.00 s, 2.00 s, and 3.00 s after it is thrown, neglecting the effects of air resistance.

### Strategy

Draw a sketch.



We are asked to determine the position  $y$  at various times. It is reasonable to take the initial position  $y_0$  to be zero. This problem involves one-dimensional motion in the vertical direction. We use plus and minus signs to indicate direction, with up being positive and down negative. Since up is positive, and the rock is thrown upward, the initial velocity must be positive too. The acceleration due to gravity is downward, so  $a$  is negative. It is crucial that the initial velocity and the acceleration due to gravity have opposite signs.

Opposite signs indicate that the acceleration due to gravity opposes the initial motion and will slow and eventually reverse it.

Since we are asked for values of position and velocity at three times, we will refer to these as  $y_1$  and  $v_1$ ;  $y_2$  and  $v_2$ ; and  $y_3$  and  $v_3$ .

#### Solution for Position $y_1$

1. Identify the knowns. We know that  $y_0 = 0$ ;  $v_0 = 13.0 \text{ m/s}$ ;

$a = -g = -9.80 \text{ m/s}^2$ ; and  $t = 1.00 \text{ s}$ .

2. Identify the best equation to use. We will use  $y = y_0 + v_0 t + \frac{1}{2} a t^2$  because it includes only one unknown,  $y$  (or  $y_1$ , here), which is the value we want to find.

3. Plug in the known values and solve for  $y_1$ .

#### Equation:

$$y_1 = 0 + (13.0 \text{ m/s})(1.00 \text{ s}) + \frac{1}{2} (-9.80 \text{ m/s}^2)(1.00 \text{ s})^2 = 8.10 \text{ m}$$

#### Discussion

The rock is 8.10 m above its starting point at  $t = 1.00 \text{ s}$ , since  $y_1 > y_0$ . It could be *moving* up or down; the only way to tell is to calculate  $v_1$  and find out if it is positive or negative.

#### Solution for Velocity $v_1$

1. Identify the knowns. We know that  $y_0 = 0$ ;  $v_0 = 13.0 \text{ m/s}$ ;

$a = -g = -9.80 \text{ m/s}^2$ ; and  $t = 1.00 \text{ s}$ . We also know from the solution above that  $y_1 = 8.10 \text{ m}$ .

2. Identify the best equation to use. The most straightforward is  $v = v_0 - gt$  (from  $v = v_0 + at$ , where  $a = \text{gravitational acceleration} = -g$ ).
3. Plug in the knowns and solve.

**Equation:**

$$v_1 = v_0 - gt = 13.0 \text{ m/s} - (9.80 \text{ m/s}^2)(1.00 \text{ s}) = 3.20 \text{ m/s}$$

### Discussion

The positive value for  $v_1$  means that the rock is still heading upward at  $t = 1.00 \text{ s}$ . However, it has slowed from its original  $13.0 \text{ m/s}$ , as expected.

### Solution for Remaining Times

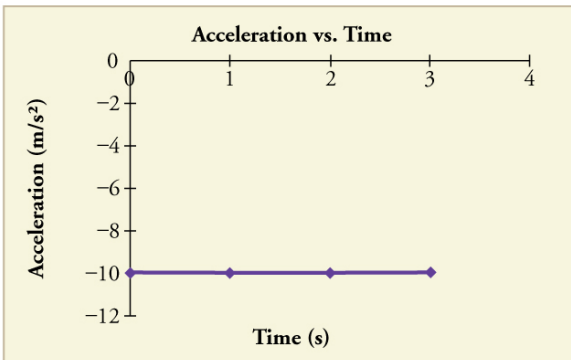
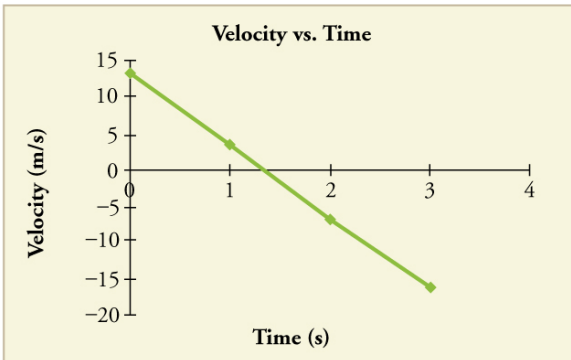
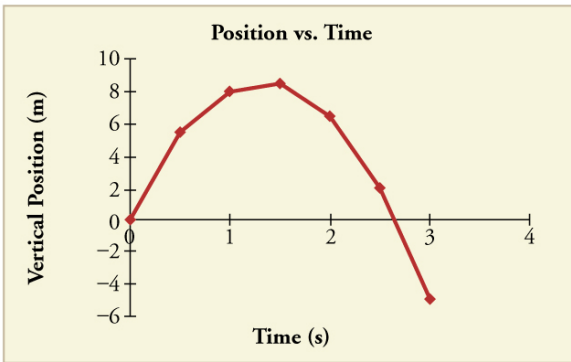
The procedures for calculating the position and velocity at  $t = 2.00 \text{ s}$  and  $3.00 \text{ s}$  are the same as those above. The results are summarized in [\[link\]](#) and illustrated in [\[link\]](#).

Time, $t$	Position, $y$	Velocity, $v$	Acceleration, $a$
1.00 s	8.10 m	3.20 m/s	$-9.80 \text{ m/s}^2$
2.00 s	6.40 m	$-6.60 \text{ m/s}$	$-9.80 \text{ m/s}^2$
3.00 s	$-5.10 \text{ m}$	$-16.4 \text{ m/s}$	$-9.80 \text{ m/s}^2$

### Results

Graphing the data helps us understand it more clearly.





Vertical position, vertical velocity, and vertical acceleration vs. time for a rock thrown vertically up at the edge of a cliff. Notice that velocity changes linearly with time and that acceleration is constant.

*Misconception Alert!* Notice that the position vs. time graph shows vertical position only. It is easy to get the impression that the graph shows some

horizontal motion—the shape of the graph looks like the path of a projectile. But this is not the case; the horizontal axis is *time*, not space. The actual path of the rock in space is straight up, and straight down.

### Discussion

The interpretation of these results is important. At 1.00 s the rock is above its starting point and heading upward, since  $y_1$  and  $v_1$  are both positive. At 2.00 s, the rock is still above its starting point, but the negative velocity means it is moving downward. At 3.00 s, both  $y_3$  and  $v_3$  are negative, meaning the rock is below its starting point and continuing to move downward. Notice that when the rock is at its highest point (at 1.5 s), its velocity is zero, but its acceleration is still  $-9.80 \text{ m/s}^2$ . Its acceleration is  $-9.80 \text{ m/s}^2$  for the whole trip—while it is moving up and while it is moving down. Note that the values for  $y$  are the positions (or displacements) of the rock, not the total distances traveled. Finally, note that free-fall applies to upward motion as well as downward. Both have the same acceleration—the acceleration due to gravity, which remains constant the entire time. Astronauts training in the famous Vomit Comet, for example, experience free-fall while arcing up as well as down, as we will discuss in more detail later.

### Note:

#### Making Connections: Take-Home Experiment—Reaction Time

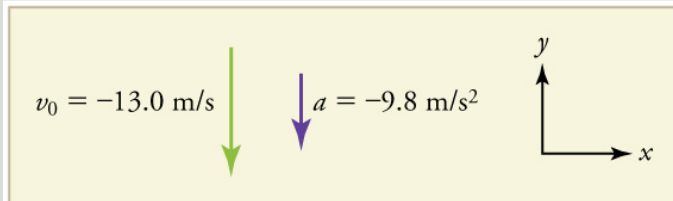
A simple experiment can be done to determine your reaction time. Have a friend hold a ruler between your thumb and index finger, separated by about 1 cm. Note the mark on the ruler that is right between your fingers. Have your friend drop the ruler unexpectedly, and try to catch it between your two fingers. Note the new reading on the ruler. Assuming acceleration is that due to gravity, calculate your reaction time. How far would you travel in a car (moving at 30 m/s) if the time it took your foot to go from the gas pedal to the brake was twice this reaction time?

**Example:****Calculating Velocity of a Falling Object: A Rock Thrown Down**

What happens if the person on the cliff throws the rock straight down, instead of straight up? To explore this question, calculate the velocity of the rock when it is 5.10 m below the starting point, and has been thrown downward with an initial speed of 13.0 m/s.

**Strategy**

Draw a sketch.



Since up is positive, the final position of the rock will be negative because it finishes below the starting point at  $y_0 = 0$ . Similarly, the initial velocity is downward and therefore negative, as is the acceleration due to gravity. We expect the final velocity to be negative since the rock will continue to move downward.

**Solution**

1. Identify the knowns.  $y_0 = 0$ ;  $y_1 = -5.10$  m;  $v_0 = -13.0$  m/s;  $a = -g = -9.80$  m/s<sup>2</sup>.
2. Choose the kinematic equation that makes it easiest to solve the problem. The equation  $v^2 = v_0^2 + 2a(y - y_0)$  works well because the only unknown in it is  $v$ . (We will plug  $y_1$  in for  $y$ .)
3. Enter the known values

**Equation:**

$$v^2 = (-13.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)(-5.10 \text{ m} - 0 \text{ m}) = 268.96 \text{ m}^2/\text{s}^2,$$

where we have retained extra significant figures because this is an intermediate result.

Taking the square root, and noting that a square root can be positive or negative, gives

**Equation:**

$$v = \pm 16.4 \text{ m/s}.$$

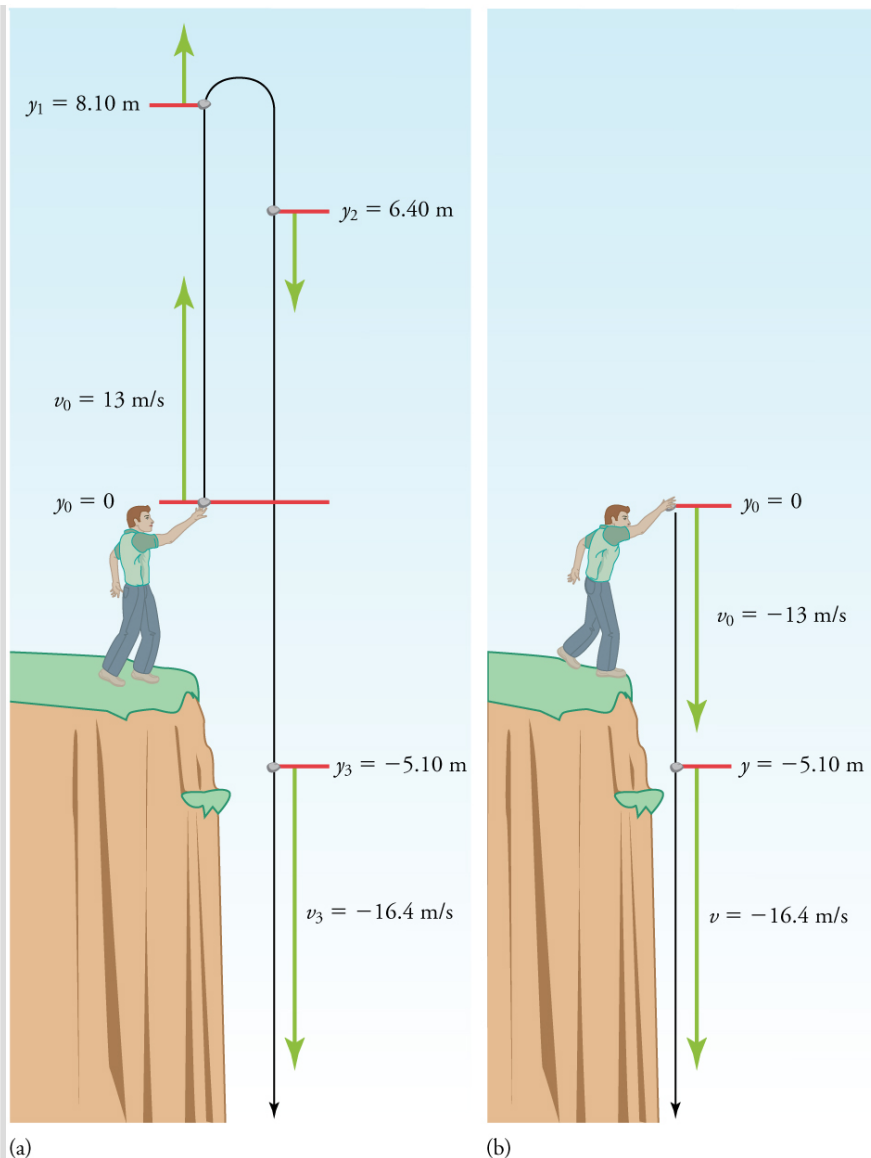
The negative root is chosen to indicate that the rock is still heading down. Thus,

**Equation:**

$$v = -16.4 \text{ m/s.}$$

### Discussion

Note that *this is exactly the same velocity the rock had at this position when it was thrown straight upward with the same initial speed.* (See [\[link\]](#) and [\[link\]](#)(a).) This is not a coincidental result. Because we only consider the acceleration due to gravity in this problem, the *speed* of a falling object depends only on its initial speed and its vertical position relative to the starting point. For example, if the velocity of the rock is calculated at a height of 8.10 m above the starting point (using the method from [\[link\]](#)) when the initial velocity is 13.0 m/s straight up, a result of  $\pm 3.20 \text{ m/s}$  is obtained. Here both signs are meaningful; the positive value occurs when the rock is at 8.10 m and heading up, and the negative value occurs when the rock is at 8.10 m and heading back down. It has the same *speed* but the opposite direction.



(a) A person throws a rock straight up, as explored in [\[link\]](#). The arrows are velocity vectors at 0, 1.00, 2.00, and 3.00 s. (b) A person throws a rock straight down from a cliff with the same initial speed as before, as in [\[link\]](#). Note that at the same distance below the point of release, the rock has the same velocity in both cases.

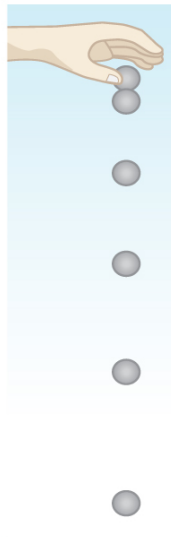
Another way to look at it is this: In [\[link\]](#), the rock is thrown up with an initial velocity of  $13.0 \text{ m/s}$ . It rises and then falls back down. When its

position is  $y = 0$  on its way back down, its velocity is  $-13.0 \text{ m/s}$ . That is, it has the same speed on its way down as on its way up. We would then expect its velocity at a position of  $y = -5.10 \text{ m}$  to be the same whether we have thrown it upwards at  $+13.0 \text{ m/s}$  or thrown it downwards at  $-13.0 \text{ m/s}$ . The velocity of the rock on its way down from  $y = 0$  is the same whether we have thrown it up or down to start with, as long as the speed with which it was initially thrown is the same.

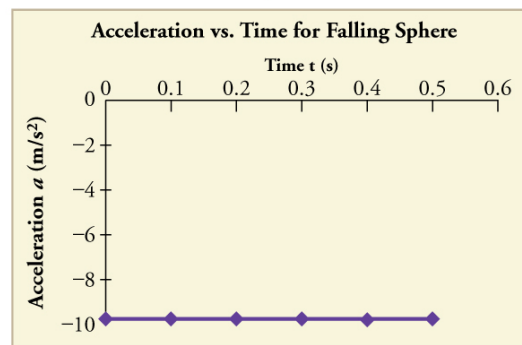
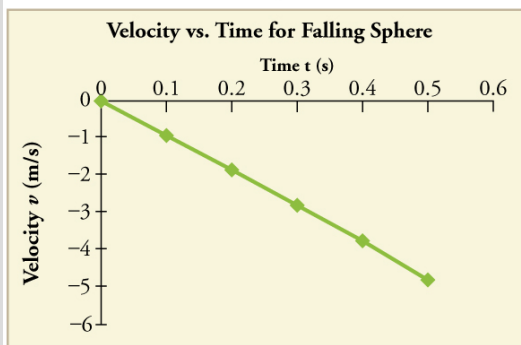
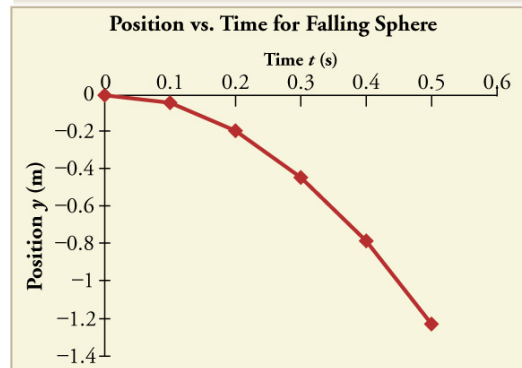
**Example:****Find  $g$  from Data on a Falling Object**

The acceleration due to gravity on Earth differs slightly from place to place, depending on topography (e.g., whether you are on a hill or in a valley) and subsurface geology (whether there is dense rock like iron ore as opposed to light rock like salt beneath you.) The precise acceleration due to gravity can be calculated from data taken in an introductory physics laboratory course.

An object, usually a metal ball for which air resistance is negligible, is dropped and the time it takes to fall a known distance is measured. See, for example, [\[link\]](#). Very precise results can be produced with this method if sufficient care is taken in measuring the distance fallen and the elapsed time.



$y$ (m)	$v$ (m/s)	$t$ (s)
0	0	0
-0.049	-0.98	0.1
-0.196	-1.96	0.2
-0.441	-2.94	0.3
-0.784	-3.92	0.4
-1.225	-4.90	0.5



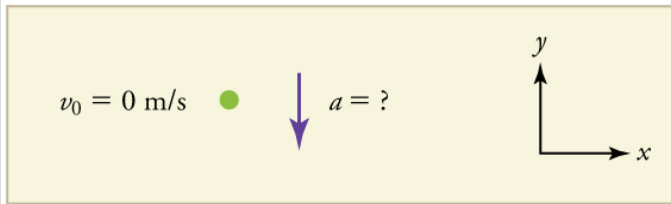
Positions and velocities of a metal ball released from rest when air resistance is negligible. Velocity is seen to increase linearly with time while displacement increases with time squared.

Acceleration is a constant and is equal to gravitational acceleration.

Suppose the ball falls 1.0000 m in 0.45173 s. Assuming the ball is not affected by air resistance, what is the precise acceleration due to gravity at this location?

**Strategy**

Draw a sketch.



We need to solve for acceleration  $a$ . Note that in this case, displacement is downward and therefore negative, as is acceleration.

**Solution**

1. Identify the knowns.  $y_0 = 0$ ;  $y = -1.0000$  m;  $t = 0.45173$ ;  $v_0 = 0$ .
2. Choose the equation that allows you to solve for  $a$  using the known values.

**Equation:**

$$y = y_0 + v_0 t + \frac{1}{2} a t^2$$

3. Substitute 0 for  $v_0$  and rearrange the equation to solve for  $a$ . Substituting 0 for  $v_0$  yields

**Equation:**

$$y = y_0 + \frac{1}{2} a t^2.$$

Solving for  $a$  gives

**Equation:**

$$a = \frac{2(y - y_0)}{t^2}.$$

4. Substitute known values yields

**Equation:**

$$a = \frac{2(-1.0000 \text{ m} - 0)}{(0.45173 \text{ s})^2} = -9.8010 \text{ m/s}^2,$$



so, because  $a = -g$  with the directions we have chosen,

**Equation:**

$$g = 9.8010 \text{ m/s}^2.$$

### Discussion

The negative value for  $a$  indicates that the gravitational acceleration is downward, as expected. We expect the value to be somewhere around the average value of  $9.80 \text{ m/s}^2$ , so  $9.8010 \text{ m/s}^2$  makes sense. Since the data going into the calculation are relatively precise, this value for  $g$  is more precise than the average value of  $9.80 \text{ m/s}^2$ ; it represents the local value for the acceleration due to gravity.

### Exercise:

#### Check Your Understanding

##### Problem:

A chunk of ice breaks off a glacier and falls 30.0 meters before it hits the water. Assuming it falls freely (there is no air resistance), how long does it take to hit the water?

---

##### Solution:

We know that initial position  $y_0 = 0$ , final position  $y = -30.0 \text{ m}$ , and  $a = -g = -9.80 \text{ m/s}^2$ . We can then use the equation

$y = y_0 + v_0t + \frac{1}{2}at^2$  to solve for  $t$ . Inserting  $a = -g$ , we obtain

**Equation:**

$$y = 0 + 0 - \frac{1}{2}gt^2$$

$$t^2 = \frac{2y}{-g}$$

$$t = \pm \sqrt{\frac{2y}{-g}} = \pm \sqrt{\frac{2(-30.0 \text{ m})}{-9.80 \text{ m/s}^2}} = \pm \sqrt{6.12 \text{ s}^2} = 2.47 \text{ s} \approx 2.5 \text{ s}$$

where we take the positive value as the physically relevant answer. Thus, it takes about 2.5 seconds for the piece of ice to hit the water.

**Note:**

**PhET Explorations: Equation Grapher**

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g.  $y = bx$ ) to see how they add to generate the polynomial curve.

[https://phet.colorado.edu/sims/equation-grapher/equation-grapher\\_en.html](https://phet.colorado.edu/sims/equation-grapher/equation-grapher_en.html)

## Section Summary

- An object in free-fall experiences constant acceleration if air resistance is negligible.
- On Earth, all free-falling objects have an acceleration due to gravity  $g$ , which averages

**Equation:**

$$g = 9.80 \text{ m/s}^2.$$

- Whether the acceleration  $a$  should be taken as  $+g$  or  $-g$  is determined by your choice of coordinate system. If you choose the upward direction as positive,  $a = -g = -9.80 \text{ m/s}^2$  is negative. In the opposite case,  $a = +g = 9.80 \text{ m/s}^2$  is positive. Since acceleration is constant, the kinematic equations above can be applied with the appropriate  $+g$  or  $-g$  substituted for  $a$ .
- For objects in free-fall, up is normally taken as positive for displacement, velocity, and acceleration.

## Conceptual Questions

**Exercise:**

**Problem:**

What is the acceleration of a rock thrown straight upward on the way up? At the top of its flight? On the way down?

**Exercise:****Problem:**

An object that is thrown straight up falls back to Earth. This is one-dimensional motion. (a) When is its velocity zero? (b) Does its velocity change direction? (c) Does the acceleration due to gravity have the same sign on the way up as on the way down?

**Exercise:****Problem:**

Suppose you throw a rock nearly straight up at a coconut in a palm tree, and the rock misses on the way up but hits the coconut on the way down. Neglecting air resistance, how does the speed of the rock when it hits the coconut on the way down compare with what it would have been if it had hit the coconut on the way up? Is it more likely to dislodge the coconut on the way up or down? Explain.

**Exercise:****Problem:**

If an object is thrown straight up and air resistance is negligible, then its speed when it returns to the starting point is the same as when it was released. If air resistance were not negligible, how would its speed upon return compare with its initial speed? How would the maximum height to which it rises be affected?

**Exercise:****Problem:**

The severity of a fall depends on your speed when you strike the ground. All factors but the acceleration due to gravity being the same, how many times higher could a safe fall on the Moon be than on Earth (gravitational acceleration on the Moon is about  $1/6$  that of the Earth)?

**Exercise:****Problem:**

How many times higher could an astronaut jump on the Moon than on Earth if his takeoff speed is the same in both locations (gravitational acceleration on the Moon is about  $1/6$  of  $g$  on Earth)?

**Problems & Exercises**

Assume air resistance is negligible unless otherwise stated.

**Exercise:****Problem:**

Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, and (d) 2.00 s for a ball thrown straight up with an initial velocity of 15.0 m/s. Take the point of release to be  $y_0 = 0$ .

---

**Solution:**

(a)  $y_1 = 6.28 \text{ m}$ ;  $v_1 = 10.1 \text{ m/s}$

(b)  $y_2 = 10.1 \text{ m}$ ;  $v_2 = 5.20 \text{ m/s}$

(c)  $y_3 = 11.5 \text{ m}$ ;  $v_3 = 0.300 \text{ m/s}$

(d)  $y_4 = 10.4 \text{ m}$ ;  $v_4 = -4.60 \text{ m/s}$

**Exercise:****Problem:**

Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, (d) 2.00, and (e) 2.50 s for a rock thrown straight down with an initial velocity of 14.0 m/s from the Verrazano Narrows Bridge in New York City. The roadway of this bridge is 70.0 m above the water.

**Exercise:**

**Problem:**

A basketball referee tosses the ball straight up for the starting tip-off. At what velocity must a basketball player leave the ground to rise 1.25 m above the floor in an attempt to get the ball?

---

**Solution:**

$$v_0 = 4.95 \text{ m/s}$$

**Exercise:****Problem:**

A rescue helicopter is hovering over a person whose boat has sunk. One of the rescuers throws a life preserver straight down to the victim with an initial velocity of 1.40 m/s and observes that it takes 1.8 s to reach the water. (a) List the knowns in this problem. (b) How high above the water was the preserver released? Note that the downdraft of the helicopter reduces the effects of air resistance on the falling life preserver, so that an acceleration equal to that of gravity is reasonable.

**Exercise:****Problem:**

A dolphin in an aquatic show jumps straight up out of the water at a velocity of 13.0 m/s. (a) List the knowns in this problem. (b) How high does his body rise above the water? To solve this part, first note that the final velocity is now a known and identify its value. Then identify the unknown, and discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking units, and discuss whether the answer is reasonable. (c) How long is the dolphin in the air? Neglect any effects due to his size or orientation.

---

**Solution:**

$$(a) a = -9.80 \text{ m/s}^2; v_0 = 13.0 \text{ m/s}; y_0 = 0 \text{ m}$$

(b)  $v = 0 \text{ m/s}$ . Unknown is distance  $y$  to top of trajectory, where velocity is zero. Use equation  $v^2 = v_0^2 + 2a(y - y_0)$  because it contains all known values except for  $y$ , so we can solve for  $y$ . Solving for  $y$  gives

**Equation:**

$$v^2 - v_0^2 = 2a(y - y_0)$$

$$\frac{v^2 - v_0^2}{2a} = y - y_0$$

$$y = y_0 + \frac{v^2 - v_0^2}{2a} = 0 \text{ m} + \frac{(0 \text{ m/s})^2 - (13.0 \text{ m/s})^2}{2(-9.80 \text{ m/s}^2)} = 8.62 \text{ m}$$

Dolphins measure about 2 meters long and can jump several times their length out of the water, so this is a reasonable result.

(c) 2.65 s

**Exercise:**

**Problem:**

A swimmer bounces straight up from a diving board and falls feet first into a pool. She starts with a velocity of 4.00 m/s, and her takeoff point is 1.80 m above the pool. (a) How long are her feet in the air? (b) What is her highest point above the board? (c) What is her velocity when her feet hit the water?

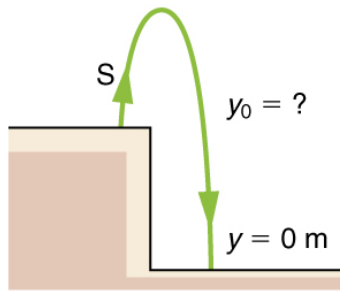
**Exercise:**

**Problem:**

(a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

---

**Solution:**



(a) 8.26 m

(b) 0.717 s

**Exercise:**

**Problem:**

A very strong, but inept, shot putter puts the shot straight up vertically with an initial velocity of 11.0 m/s. How long does he have to get out of the way if the shot was released at a height of 2.20 m, and he is 1.80 m tall?

**Exercise:**

**Problem:**

You throw a ball straight up with an initial velocity of 15.0 m/s. It passes a tree branch on the way up at a height of 7.00 m. How much additional time will pass before the ball passes the tree branch on the way back down?

---

**Solution:**

1.91 s

**Exercise:**

**Problem:**

A kangaroo can jump over an object 2.50 m high. (a) Calculate its vertical speed when it leaves the ground. (b) How long is it in the air?

**Exercise:**

**Problem:**

Standing at the base of one of the cliffs of Mt. Arapiles in Victoria, Australia, a hiker hears a rock break loose from a height of 105 m. He can't see the rock right away but then does, 1.50 s later. (a) How far above the hiker is the rock when he can see it? (b) How much time does he have to move before the rock hits his head?

---

**Solution:**

(a) 94.0 m

(b) 3.13 s

**Exercise:****Problem:**

An object is dropped from a height of 75.0 m above ground level. (a) Determine the distance traveled during the first second. (b) Determine the final velocity at which the object hits the ground. (c) Determine the distance traveled during the last second of motion before hitting the ground.

**Exercise:****Problem:**

There is a 250-m-high cliff at Half Dome in Yosemite National Park in California. Suppose a boulder breaks loose from the top of this cliff. (a) How fast will it be going when it strikes the ground? (b) Assuming a reaction time of 0.300 s, how long will a tourist at the bottom have to get out of the way after hearing the sound of the rock breaking loose (neglecting the height of the tourist, which would become negligible anyway if hit)? The speed of sound is 335 m/s on this day.

---

**Solution:**

(a) -70.0 m/s (downward)

(b) 6.10 s



**Exercise:****Problem:**

A ball is thrown straight up. It passes a 2.00-m-high window 7.50 m off the ground on its path up and takes 0.312 s to go past the window. What was the ball's initial velocity? Hint: First consider only the distance along the window, and solve for the ball's velocity at the bottom of the window. Next, consider only the distance from the ground to the bottom of the window, and solve for the initial velocity using the velocity at the bottom of the window as the final velocity.

**Exercise:****Problem:**

Suppose you drop a rock into a dark well and, using precision equipment, you measure the time for the sound of a splash to return. (a) Neglecting the time required for sound to travel up the well, calculate the distance to the water if the sound returns in 2.0000 s. (b) Now calculate the distance taking into account the time for sound to travel up the well. The speed of sound is 332.00 m/s in this well.

---

**Solution:**

(a) 19.6 m

(b) 18.5 m

**Exercise:****Problem:**

A steel ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.45 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 0.0800 ms ( $8.00 \times 10^{-5}$  s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

**Exercise:**

**Problem:**

A coin is dropped from a hot-air balloon that is 300 m above the ground and rising at 10.0 m/s upward. For the coin, find (a) the maximum height reached, (b) its position and velocity 4.00 s after being released, and (c) the time before it hits the ground.

---

**Solution:**

(a) 305 m

(b) 262 m, -29.2 m/s

(c) 8.91 s

**Exercise:****Problem:**

A soft tennis ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.10 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 3.50 ms ( $3.50 \times 10^{-3}$  s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

**Glossary**

free-fall

the state of movement that results from gravitational force only

acceleration due to gravity

acceleration of an object as a result of gravity

## Graphical Analysis of One-Dimensional Motion

- Describe a straight-line graph in terms of its slope and y-intercept.
- Determine average velocity or instantaneous velocity from a graph of position vs. time.
- Determine average or instantaneous acceleration from a graph of velocity vs. time.
- Derive a graph of velocity vs. time from a graph of position vs. time.
- Derive a graph of acceleration vs. time from a graph of velocity vs. time.

A graph, like a picture, is worth a thousand words. Graphs not only contain numerical information; they also reveal relationships between physical quantities. This section uses graphs of position, velocity, and acceleration versus time to illustrate one-dimensional kinematics.

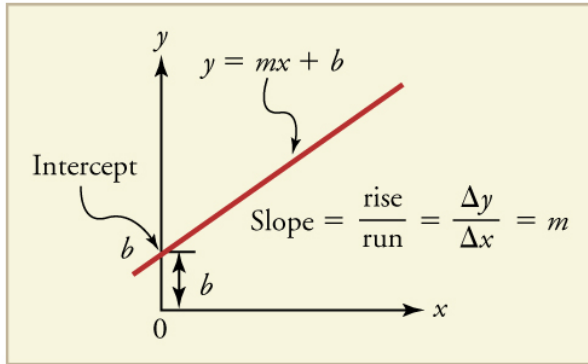
## Slopes and General Relationships

First note that graphs in this text have perpendicular axes, one horizontal and the other vertical. When two physical quantities are plotted against one another in such a graph, the horizontal axis is usually considered to be an **independent variable** and the vertical axis a **dependent variable**. If we call the horizontal axis the  $x$ -axis and the vertical axis the  $y$ -axis, as in [\[link\]](#), a straight-line graph has the general form

**Equation:**

$$y = mx + b.$$

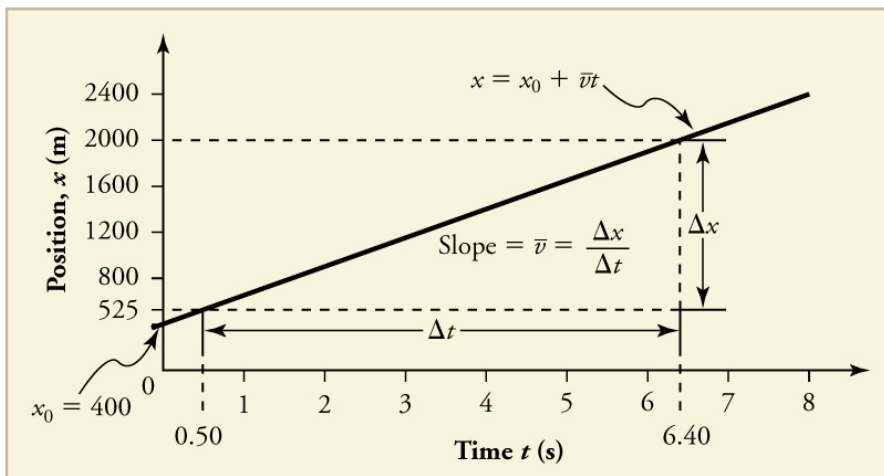
Here  $m$  is the **slope**, defined to be the rise divided by the run (as seen in the figure) of the straight line. The letter  $b$  is used for the **y-intercept**, which is the point at which the line crosses the vertical axis.



A straight-line graph. The equation for a straight line is  $y = mx + b$ .

### Graph of Position vs. Time ( $a = 0$ , so $v$ is constant)

Time is usually an independent variable that other quantities, such as position, depend upon. A graph of position versus time would, thus, have  $x$  on the vertical axis and  $t$  on the horizontal axis. [\[link\]](#) is just such a straight-line graph. It shows a graph of position versus time for a jet-powered car on a very flat dry lake bed in Nevada.



## Graph of position versus time for a jet-powered car on the Bonneville Salt Flats.

Using the relationship between dependent and independent variables, we see that the slope in the graph above is average velocity  $\bar{v}$  and the intercept is position at time zero—that is,  $x_0$ . Substituting these symbols into  $y = mx + b$  gives

**Equation:**

$$x = \bar{v}t + x_0$$

or

**Equation:**

$$x = x_0 + \bar{v}t.$$

Thus a graph of position versus time gives a general relationship among displacement(change in position), velocity, and time, as well as giving detailed numerical information about a specific situation.

**Note:**

The Slope of  $x$  vs.  $t$

The slope of the graph of position  $x$  vs. time  $t$  is velocity  $v$ .

**Equation:**

$$\text{slope} = \frac{\Delta x}{\Delta t} = v$$

Notice that this equation is the same as that derived algebraically from other motion equations in [Motion Equations for Constant Acceleration in One Dimension](#).

From the figure we can see that the car has a position of 25 m at 0.50 s and 2000 m at 6.40 s. Its position at other times can be read from the graph; furthermore, information about its velocity and acceleration can also be obtained from the graph.

**Example:**

**Determining Average Velocity from a Graph of Position versus Time:  
Jet Car**

Find the average velocity of the car whose position is graphed in [\[link\]](#).

**Strategy**

The slope of a graph of  $x$  vs.  $t$  is average velocity, since slope equals rise over run. In this case, rise = change in position and run = change in time, so that

**Equation:**

$$\text{slope} = \frac{\Delta x}{\Delta t} = \bar{v}.$$

Since the slope is constant here, any two points on the graph can be used to find the slope. (Generally speaking, it is most accurate to use two widely separated points on the straight line. This is because any error in reading data from the graph is proportionally smaller if the interval is larger.)

**Solution**

1. Choose two points on the line. In this case, we choose the points labeled on the graph: (6.4 s, 2000 m) and (0.50 s, 525 m). (Note, however, that you could choose any two points.)
2. Substitute the  $x$  and  $t$  values of the chosen points into the equation. Remember in calculating change ( $\Delta$ ) we always use final value minus initial value.

**Equation:**

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{2000 \text{ m} - 525 \text{ m}}{6.4 \text{ s} - 0.50 \text{ s}},$$

yielding

**Equation:**

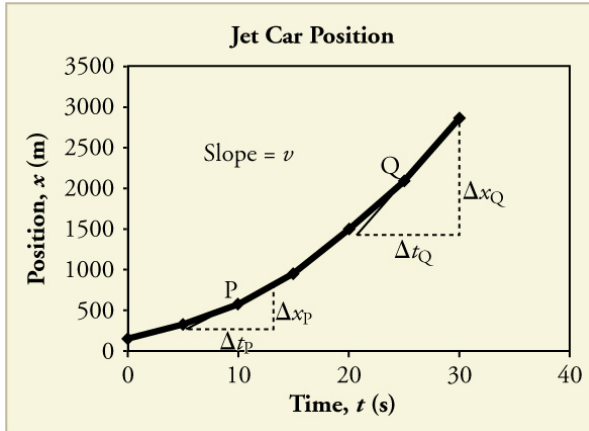
$$\bar{v} = 250 \text{ m/s.}$$

**Discussion**

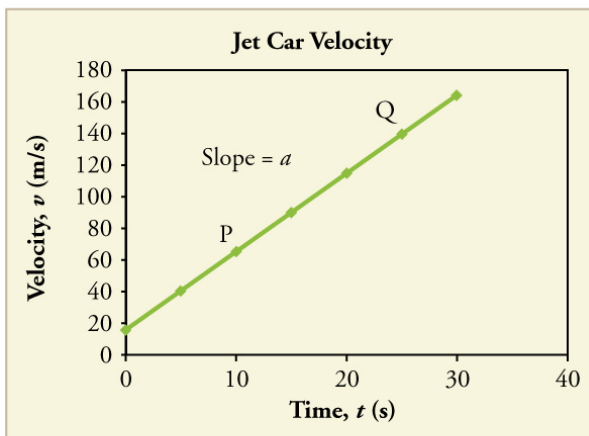
This is an impressively large land speed (900 km/h, or about 560 mi/h): much greater than the typical highway speed limit of 60 mi/h (27 m/s or 96 km/h), but considerably shy of the record of 343 m/s (1234 km/h or 766 mi/h) set in 1997.

**Graphs of Motion when  $a$  is constant but  $a \neq 0$** 

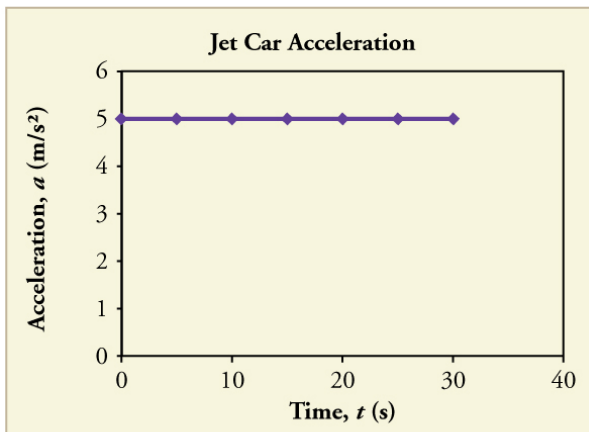
The graphs in [\[link\]](#) below represent the motion of the jet-powered car as it accelerates toward its top speed, but only during the time when its acceleration is constant. Time starts at zero for this motion (as if measured with a stopwatch), and the position and velocity are initially 200 m and 15 m/s, respectively.



(a)



(b)



(c)

Graphs of motion of a jet-powered car during the time span when its acceleration is constant. (a) The slope of an  $x$  vs.  $t$  graph is velocity. This is



shown at two points, and the instantaneous velocities obtained are plotted in the next graph. Instantaneous velocity at any point is the slope of the tangent at that point. (b) The slope of the  $v$  vs.  $t$  graph is constant for this part of the motion, indicating constant acceleration. (c) Acceleration has the constant value of  $5.0 \text{ m/s}^2$  over the time interval plotted.



A U.S. Air Force jet car speeds down a track. (credit: Matt Trostle, Flickr)

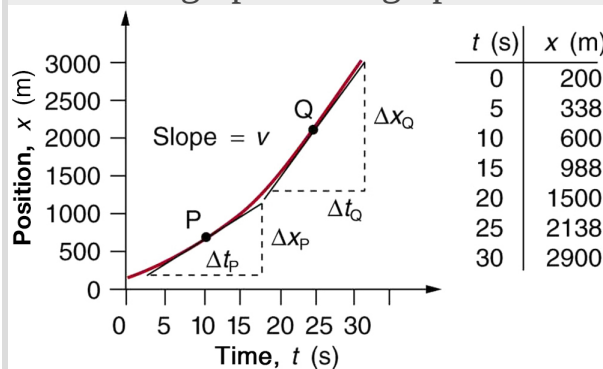
The graph of position versus time in [\[link\]](#)(a) is a curve rather than a straight line. The slope of the curve becomes steeper as time progresses,

showing that the velocity is increasing over time. The slope at any point on a position-versus-time graph is the instantaneous velocity at that point. It is found by drawing a straight line tangent to the curve at the point of interest and taking the slope of this straight line. Tangent lines are shown for two points in [\[link\]\(a\)](#). If this is done at every point on the curve and the values are plotted against time, then the graph of velocity versus time shown in [\[link\]\(b\)](#) is obtained. Furthermore, the slope of the graph of velocity versus time is acceleration, which is shown in [\[link\]\(c\)](#).

### Example:

#### Determining Instantaneous Velocity from the Slope at a Point: Jet Car

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the  $x$  vs.  $t$  graph in the graph below.



The slope of an  $x$  vs.  $t$  graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.

### Strategy

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in [\[link\]](#), where Q is the point at  $t = 25$  s.

### Solution

1. Find the tangent line to the curve at  $t = 25$  s.

2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.
3. Plug these endpoints into the equation to solve for the slope,  $v$ .

**Equation:**

$$\text{slope} = v_Q = \frac{\Delta x_Q}{\Delta t_Q} = \frac{(3120 \text{ m} - 1300 \text{ m})}{(32 \text{ s} - 19 \text{ s})}$$

Thus,

**Equation:**

$$v_Q = \frac{1820 \text{ m}}{13 \text{ s}} = 140 \text{ m/s}.$$

### Discussion

This is the value given in this figure's table for  $v$  at  $t = 25$  s. The value of 140 m/s for  $v_Q$  is plotted in [\[link\]](#). The entire graph of  $v$  vs.  $t$  can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a  $v$  vs.  $t$  graph, rise = change in velocity  $\Delta v$  and run = change in time  $\Delta t$ .

### Note:

The Slope of  $v$  vs.  $t$

The slope of a graph of velocity  $v$  vs. time  $t$  is acceleration  $a$ .

**Equation:**

$$\text{slope} = \frac{\Delta v}{\Delta t} = a$$

Since the velocity versus time graph in [\[link\]](#)(b) is a straight line, its slope is the same everywhere, implying that acceleration is constant. Acceleration versus time is graphed in [\[link\]](#)(c).

Additional general information can be obtained from [\[link\]](#) and the expression for a straight line,  $y = mx + b$ .

In this case, the vertical axis  $y$  is  $V$ , the intercept  $b$  is  $v_0$ , the slope  $m$  is  $a$ , and the horizontal axis  $x$  is  $t$ . Substituting these symbols yields

**Equation:**

$$v = v_0 + at.$$

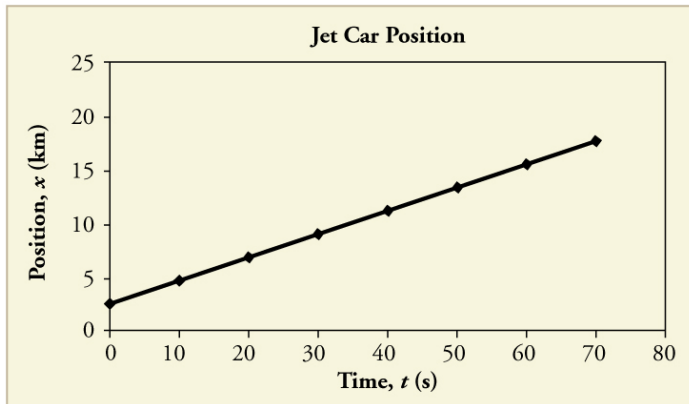
A general relationship for velocity, acceleration, and time has again been obtained from a graph. Notice that this equation was also derived algebraically from other motion equations in [Motion Equations for Constant Acceleration in One Dimension](#).

It is not accidental that the same equations are obtained by graphical analysis as by algebraic techniques. In fact, an important way to *discover* physical relationships is to measure various physical quantities and then make graphs of one quantity against another to see if they are correlated in any way. Correlations imply physical relationships and might be shown by smooth graphs such as those above. From such graphs, mathematical relationships can sometimes be postulated. Further experiments are then performed to determine the validity of the hypothesized relationships.

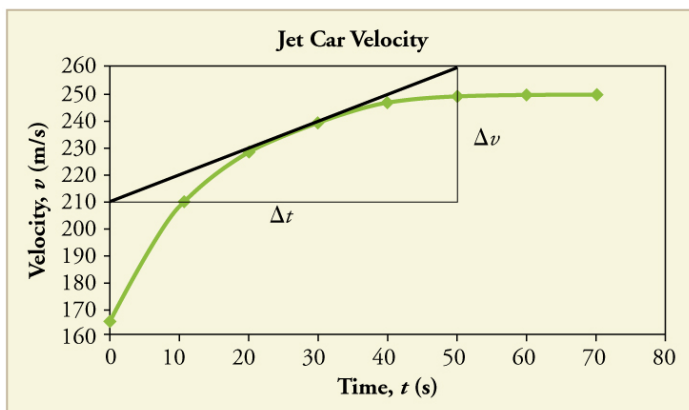
## Graphs of Motion Where Acceleration is Not Constant

Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in [\[link\]](#). Time again starts at zero, and the initial position and velocity are 2900 m and 165 m/s, respectively. (These were the final position and velocity of the car in the motion graphed in [\[link\]](#).) Acceleration gradually decreases from  $5.0 \text{ m/s}^2$  to zero when the car hits 250 m/s. The slope of the  $x$  vs.  $t$  graph increases until  $t = 55 \text{ s}$ , after which time the slope is constant. Similarly, velocity increases until 55

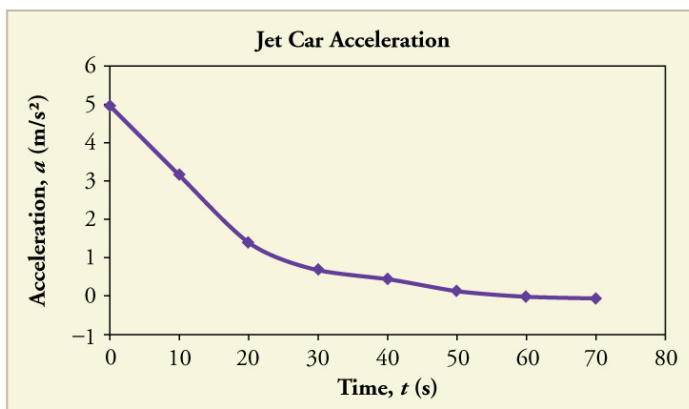
s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.



(a)



(b)



(c)

Graphs of motion of a jet-powered car as it reaches its top velocity. This motion begins where the motion in

[\[link\]](#) ends. (a) The slope of this graph is velocity; it is plotted in the next graph. (b) The velocity gradually approaches its top value. The slope of this graph is acceleration; it is plotted in the final graph. (c) Acceleration gradually declines to zero when velocity becomes constant.

**Example:****Calculating Acceleration from a Graph of Velocity versus Time**

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the  $v$  vs.  $t$  graph in [\[link\]](#)(b).

**Strategy**

The slope of the curve at  $t = 25$  s is equal to the slope of the line tangent at that point, as illustrated in [\[link\]](#)(b).

**Solution**

Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope,  $a$ .

**Equation:**

$$\text{slope} = \frac{\Delta v}{\Delta t} = \frac{(260 \text{ m/s} - 210 \text{ m/s})}{(51 \text{ s} - 1.0 \text{ s})}$$

**Equation:**

$$a = \frac{50 \text{ m/s}}{50 \text{ s}} = 1.0 \text{ m/s}^2.$$

**Discussion**

Note that this value for  $a$  is consistent with the value plotted in [\[link\]](#)(c) at  $t = 25$  s.

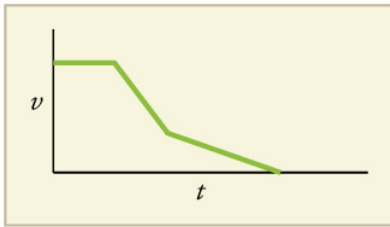
A graph of position versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point. Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.

**Exercise:**

**Check Your Understanding**

**Problem:**

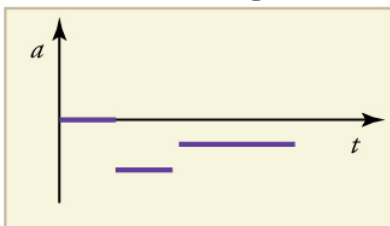
A graph of velocity vs. time of a ship coming into a harbor is shown below. (a) Describe the motion of the ship based on the graph. (b) What would a graph of the ship's acceleration look like?



**Solution:**

(a) The ship moves at constant velocity and then begins to decelerate at a constant rate. At some point, its deceleration rate decreases. It maintains this lower deceleration rate until it stops moving.

(b) A graph of acceleration vs. time would show zero acceleration in the first leg, large and constant negative acceleration in the second leg, and constant negative acceleration.



## Section Summary

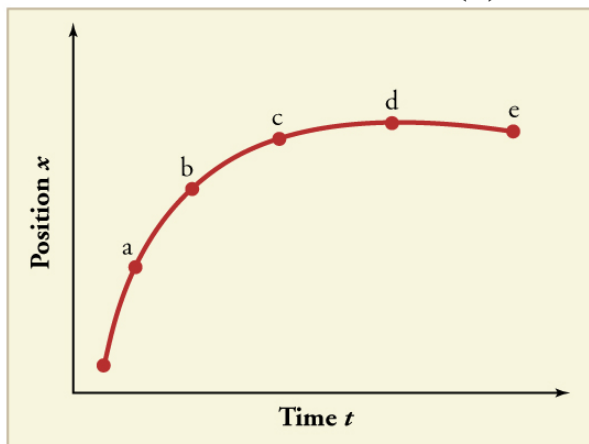
- Graphs of motion can be used to analyze motion.
- Graphical solutions yield identical solutions to mathematical methods for deriving motion equations.
- The slope of a graph of displacement  $x$  vs. time  $t$  is velocity  $v$ .
- The slope of a graph of velocity  $v$  vs. time  $t$  graph is acceleration  $a$ .
- Average velocity, instantaneous velocity, and acceleration can all be obtained by analyzing graphs.

## Conceptual Questions

### Exercise:

#### Problem:

(a) Explain how you can use the graph of position versus time in [\[link\]](#) to describe the change in velocity over time. Identify (b) the time ( $t_a$ ,  $t_b$ ,  $t_c$ ,  $t_d$ , or  $t_e$ ) at which the instantaneous velocity is greatest, (c) the time at which it is zero, and (d) the time at which it is negative.

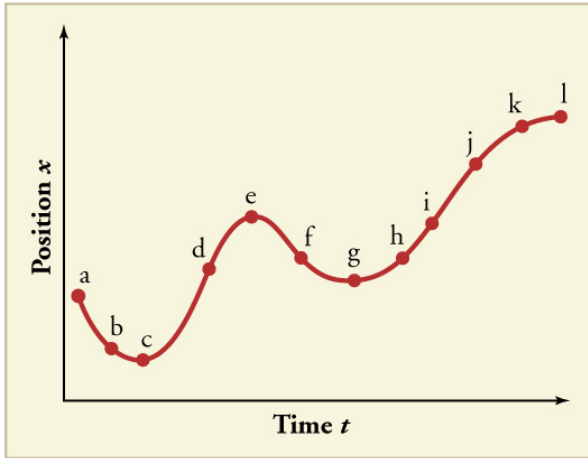


### Exercise:

#### Problem:

(a) Sketch a graph of velocity versus time corresponding to the graph of position versus time given in [\[link\]](#). (b) Identify the time or times ( $t_a$ ,  $t_b$ ,  $t_c$ , etc.) at which the instantaneous velocity is greatest. (c) At which times is it zero? (d) At which times is it negative?

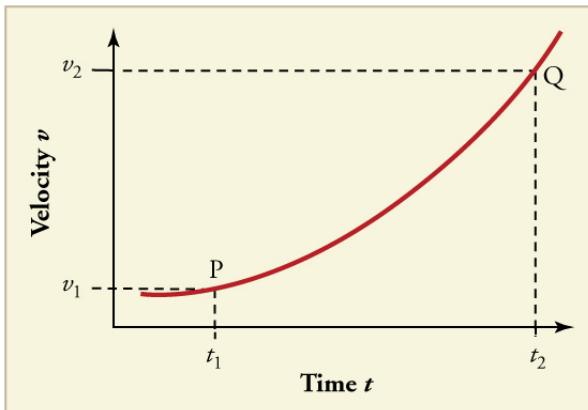




**Exercise:**

**Problem:**

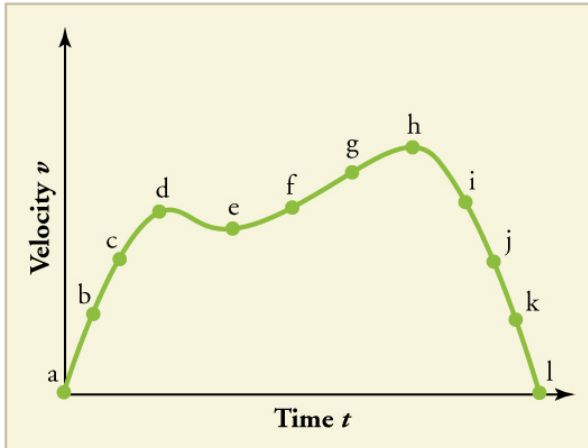
(a) Explain how you can determine the acceleration over time from a velocity versus time graph such as the one in [\[link\]](#). (b) Based on the graph, how does acceleration change over time?



**Exercise:**

**Problem:**

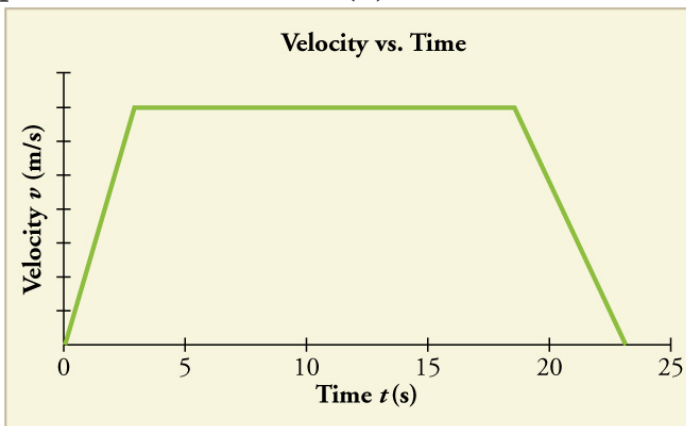
(a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in [\[link\]](#). (b) Identify the time or times ( $t_a$ ,  $t_b$ ,  $t_c$ , etc.) at which the acceleration is greatest. (c) At which times is it zero? (d) At which times is it negative?



### Exercise:

#### Problem:

Consider the velocity vs. time graph of a person in an elevator shown in [\[link\]](#). Suppose the elevator is initially at rest. It then accelerates for 3 seconds, maintains that velocity for 15 seconds, then decelerates for 5 seconds until it stops. The acceleration for the entire trip is not constant so we cannot use the equations of motion from [Motion Equations for Constant Acceleration in One Dimension](#) for the complete trip. (We could, however, use them in the three individual sections where acceleration is a constant.) Sketch graphs of (a) position vs. time and (b) acceleration vs. time for this trip.



### Exercise:

**Problem:**

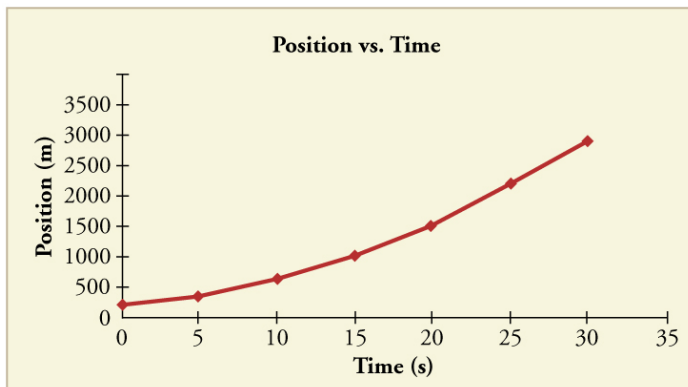
A cylinder is given a push and then rolls up an inclined plane. If the origin is the starting point, sketch the position, velocity, and acceleration of the cylinder vs. time as it goes up and then down the plane.

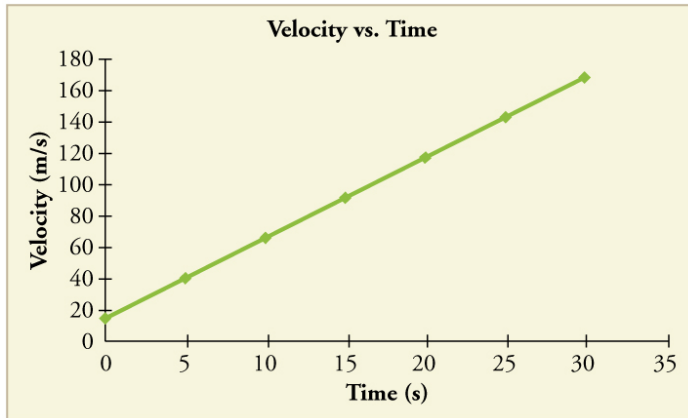
**Problems & Exercises**

Note: There is always uncertainty in numbers taken from graphs. If your answers differ from expected values, examine them to see if they are within data extraction uncertainties estimated by you.

**Exercise:****Problem:**

(a) By taking the slope of the curve in [\[link\]](#), verify that the velocity of the jet car is 115 m/s at  $t = 20$  s. (b) By taking the slope of the curve at any point in [\[link\]](#), verify that the jet car's acceleration is  $5.0 \text{ m/s}^2$ .





**Solution:**

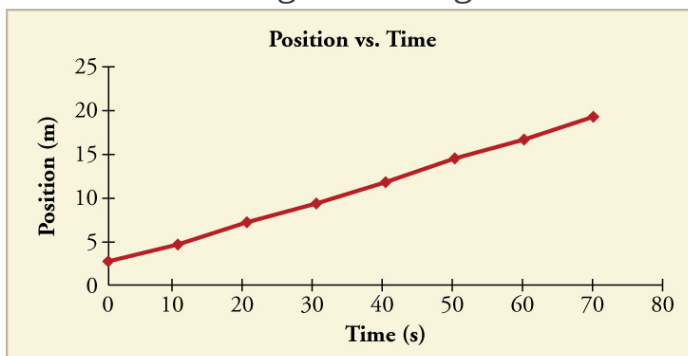
(a) 115 m/s

(b) 5.0 m/s<sup>2</sup>

**Exercise:**

**Problem:**

Using approximate values, calculate the slope of the curve in [\[link\]](#) to verify that the velocity at  $t = 10.0$  s is 0.208 m/s. Assume all values are known to 3 significant figures.



**Exercise:**

**Problem:**

Using approximate values, calculate the slope of the curve in [\[link\]](#) to verify that the velocity at  $t = 30.0$  s is approximately 0.24 m/s.

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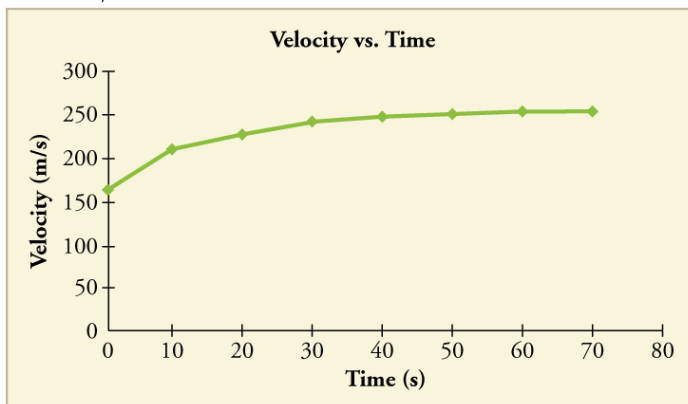
**Solution:**  
**Equation:**

$$v = \frac{(11.7 - 6.95) \times 10^3 \text{ m}}{(40.0 - 20.0) \text{ s}} = 238 \text{ m/s}$$

**Exercise:**

**Problem:**

By taking the slope of the curve in [\[link\]](#), verify that the acceleration is  $3.2 \text{ m/s}^2$  at  $t = 10 \text{ s}$ .



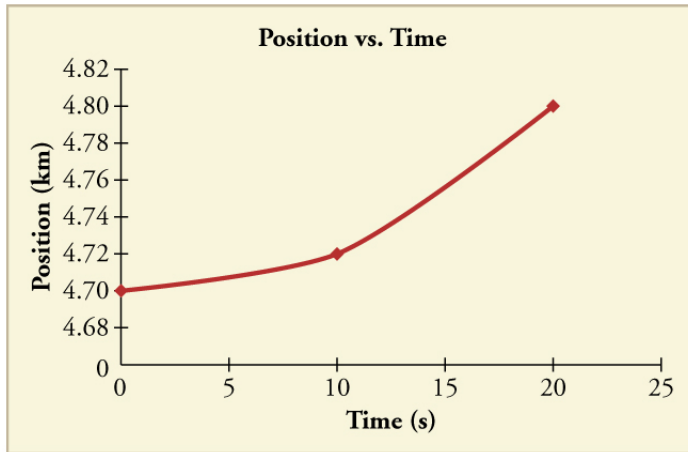
**Exercise:**

**Problem:**

Construct the position graph for the subway shuttle train as shown in [\[link\]](#)(a). Your graph should show the position of the train, in kilometers, from  $t = 0$  to  $20 \text{ s}$ . You will need to use the information on acceleration and velocity given in the examples for this figure.

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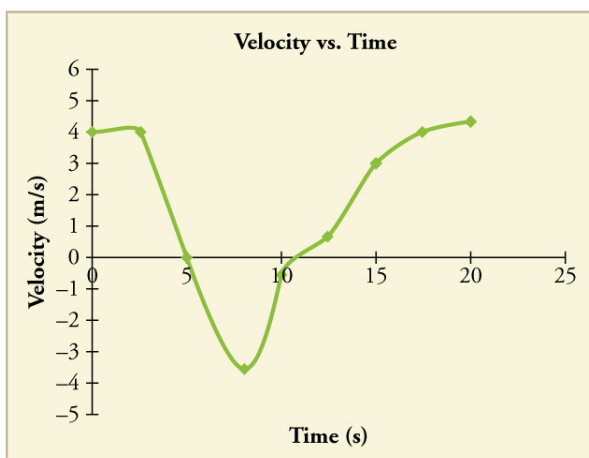
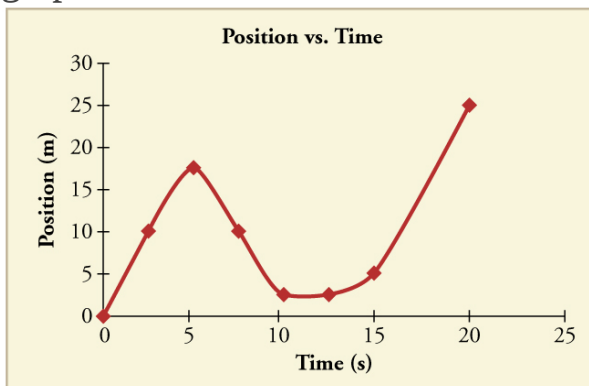
**Solution:**

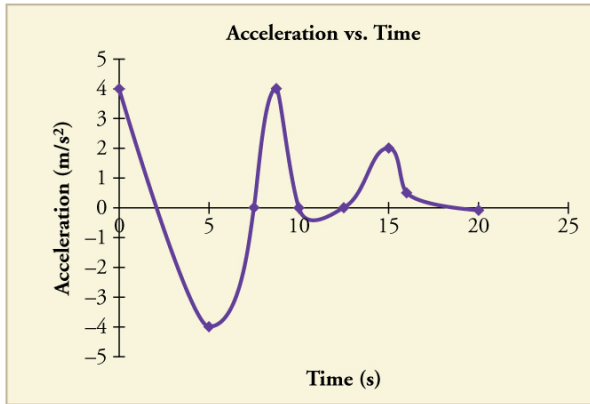


### Exercise:

#### Problem:

(a) Take the slope of the curve in [\[link\]](#) to find the jogger's velocity at  $t = 2.5$  s. (b) Repeat at 7.5 s. These values must be consistent with the graph in [\[link\]](#).

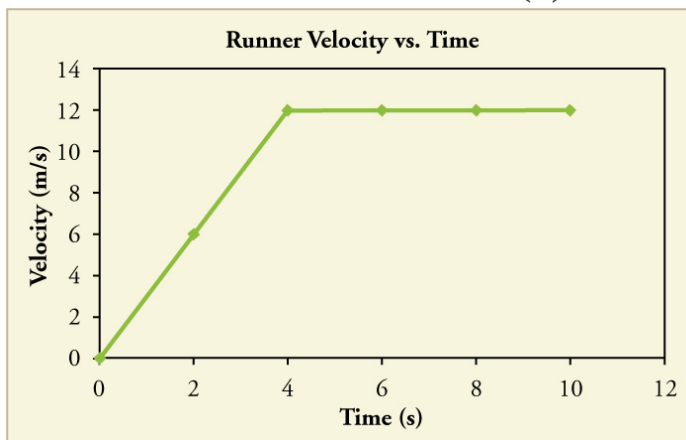




### Exercise:

#### Problem:

A graph of  $v(t)$  is shown for a world-class track sprinter in a 100-m race. (See [link](#)). (a) What is his average velocity for the first 4 s? (b) What is his instantaneous velocity at  $t = 5$  s? (c) What is his average acceleration between 0 and 4 s? (d) What is his time for the race?



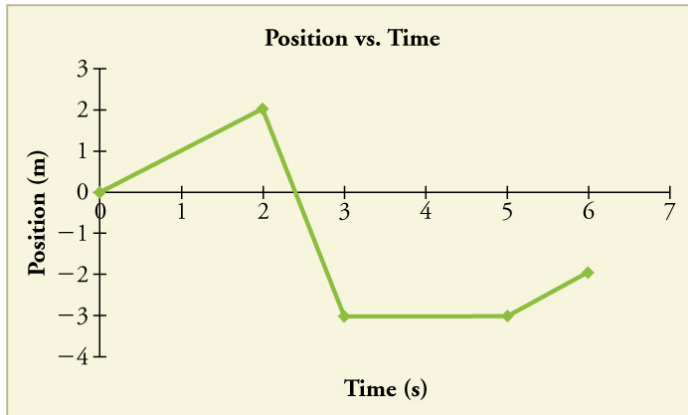
#### Solution:

- (a) 6 m/s
- (b) 12 m/s
- (c) 3  $\text{m/s}^2$
- (d) 10 s

## Exercise:

### Problem:

[\[link\]](#) shows the position graph for a particle for 5 s. Draw the corresponding velocity and acceleration graphs.



## Glossary

independent variable

the variable that the dependent variable is measured with respect to;  
usually plotted along the  $x$ -axis

dependent variable

the variable that is being measured; usually plotted along the  $y$ -axis

slope

the difference in  $y$ -value (the rise) divided by the difference in  $x$ -value (the run) of two points on a straight line

$y$ -intercept

the  $y$ -value when  $x = 0$ , or when the graph crosses the  $y$ -axis



## Introduction to Two-Dimensional Kinematics

class="introduction"

Everyday motion  
that we experience  
is, thankfully,  
rarely as tortuous  
as a rollercoaster  
ride like this—the  
Dragon Khan in  
Spain's Universal  
Port Aventura  
Amusement Park.

However, most  
motion is in  
curved, rather than  
straight-line, paths.

Motion along a  
curved path is two-  
or three-  
dimensional  
motion, and can be  
described in a  
similar fashion to  
one-dimensional  
motion. (credit:  
Boris23/Wikimedi  
a Commons)



The arc of a basketball, the orbit of a satellite, a bicycle rounding a curve, a swimmer diving into a pool, blood gushing out of a wound, and a puppy chasing its tail are but a few examples of motions along curved paths. In fact, most motions in nature follow curved paths rather than straight lines. Motion along a curved path on a flat surface or a plane (such as that of a ball on a pool table or a skater on an ice rink) is two-dimensional, and thus described by two-dimensional kinematics. Motion not confined to a plane, such as a car following a winding mountain road, is described by three-dimensional kinematics. Both two- and three-dimensional kinematics are simple extensions of the one-dimensional kinematics developed for straight-line motion in the previous chapter. This simple extension will allow us to apply physics to many more situations, and it will also yield unexpected insights about nature.

## Kinematics in Two Dimensions: An Introduction

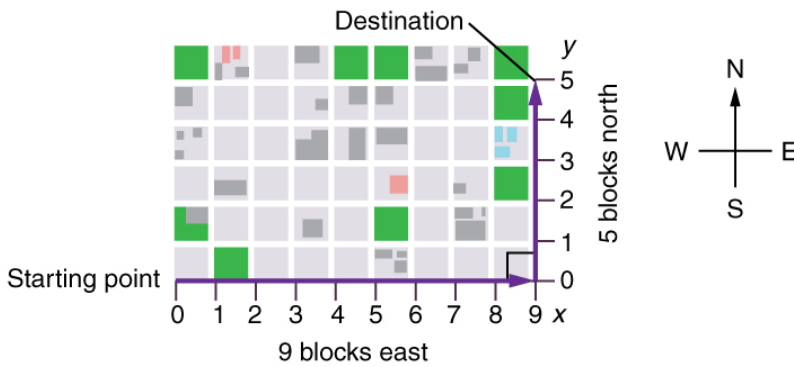
- Observe that motion in two dimensions consists of horizontal and vertical components.
- Understand the independence of horizontal and vertical vectors in two-dimensional motion.



Walkers and drivers in a city like New York are rarely able to travel in straight lines to reach their destinations. Instead, they must follow roads and sidewalks, making two-dimensional, zigzagged paths.  
(credit: Margaret W. Carruthers)

## Two-Dimensional Motion: Walking in a City


Suppose you want to walk from one point to another in a city with uniform square blocks, as pictured in [\[link\]](#).



A pedestrian walks a two-dimensional path between two points in a city. In this scene, all blocks are square and are the same size.

The straight-line path that a helicopter might fly is blocked to you as a pedestrian, and so you are forced to take a two-dimensional path, such as the one shown. You walk 14 blocks in all, 9 east followed by 5 north. What is the straight-line distance?

An old adage states that the shortest distance between two points is a straight line. The two legs of the trip and the straight-line path form a right triangle, and so the Pythagorean theorem,  $a^2 + b^2 = c^2$ , can be used to find the straight-line distance.

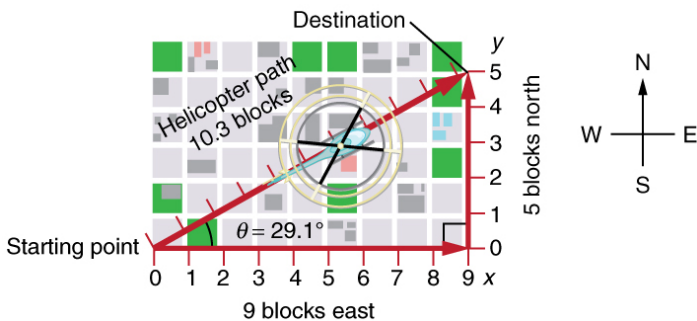


A right triangle is shown with a horizontal base labeled  $a$ , a vertical side labeled  $b$ , and a hypotenuse labeled  $c$ . To the left of the triangle, the equation  $c = \sqrt{a^2 + b^2}$  is written.

The Pythagorean theorem relates the length of the legs of a right triangle,

labeled  $a$  and  $b$ ,  
 with the  
 hypotenuse, labeled  
 $c$ . The relationship  
 is given by:  
 $a^2 + b^2 = c^2$ . This  
 can be rewritten,  
 solving for  $c$  :  
 $c = \sqrt{a^2 + b^2}$ .

The hypotenuse of the triangle is the straight-line path, and so in this case its length in units of city blocks is  
 $\sqrt{(9 \text{ blocks})^2 + (5 \text{ blocks})^2} = 10.3 \text{ blocks}$ , considerably shorter than the 14 blocks you walked. (Note that we are using three significant figures in the answer. Although it appears that “9” and “5” have only one significant digit, they are discrete numbers. In this case “9 blocks” is the same as “9.0 or 9.00 blocks.” We have decided to use three significant figures in the answer in order to show the result more precisely.)



The straight-line path followed by a helicopter between the two points is shorter than the 14 blocks walked by the pedestrian. All blocks are square and the same size.

The fact that the straight-line distance (10.3 blocks) in [\[link\]](#) is less than the total distance walked (14 blocks) is one example of a general characteristic of vectors. (Recall that **vectors** are quantities that have both magnitude and direction.)

As for one-dimensional kinematics, we use arrows to represent vectors. The length of the arrow is proportional to the vector's magnitude. The arrow's length is indicated by hash marks in [\[link\]](#) and [\[link\]](#). The arrow points in the same direction as the vector. For two-dimensional motion, the path of an object can be represented with three vectors: one vector shows the straight-line path between the initial and final points of the motion, one vector shows the horizontal component of the motion, and one vector shows the vertical component of the motion. The horizontal and vertical components of the motion add together to give the straight-line path. For example, observe the three vectors in [\[link\]](#). The first represents a 9-block displacement east. The second represents a 5-block displacement north. These vectors are added to give the third vector, with a 10.3-block total displacement. The third vector is the straight-line path between the two points. Note that in this example, the vectors that we are adding are perpendicular to each other and thus form a right triangle. This means that we can use the Pythagorean theorem to calculate the magnitude of the total displacement. (Note that we cannot use the Pythagorean theorem to add vectors that are not perpendicular. We will develop techniques for adding vectors having any direction, not just those perpendicular to one another, in [Vector Addition and Subtraction: Graphical Methods](#) and [Vector Addition and Subtraction: Analytical Methods](#).)

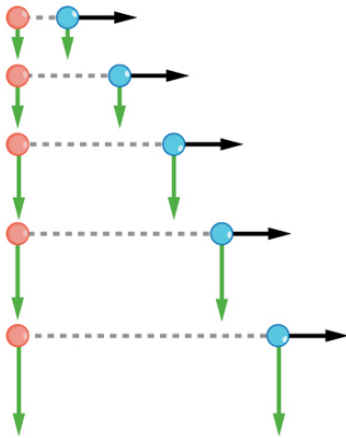
## **The Independence of Perpendicular Motions**

The person taking the path shown in [\[link\]](#) walks east and then north (two perpendicular directions). How far he or she walks east is only affected by his or her motion eastward. Similarly, how far he or she walks north is only affected by his or her motion northward.

**Note:****Independence of Motion**

The horizontal and vertical components of two-dimensional motion are independent of each other. Any motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

This is true in a simple scenario like that of walking in one direction first, followed by another. It is also true of more complicated motion involving movement in two directions at once. For example, let's compare the motions of two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and follows a curved path. A stroboscope has captured the positions of the balls at fixed time intervals as they fall.



This shows the motions of two identical balls—one falls from rest, the other has an initial horizontal velocity. Each subsequent

position is an equal time interval. Arrows represent horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity, while the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls. This shows that the vertical and horizontal motions are independent.

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies that the vertical motion is independent of whether or not the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, and not by any horizontal forces.) Careful examination of the



ball thrown horizontally shows that it travels the same horizontal distance between flashes. This is due to the fact that there are no additional forces on the ball in the horizontal direction after it is thrown. This result means that the horizontal velocity is constant, and affected neither by vertical motion nor by gravity (which is vertical). Note that this case is true only for ideal conditions. In the real world, air resistance will affect the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called *projectile motion*, is to *resolve* (break) it into motions along perpendicular directions. Resolving two-dimensional motion into perpendicular components is possible because the components are independent. We shall see how to resolve vectors in [Vector Addition and Subtraction: Graphical Methods](#) and [Vector Addition and Subtraction: Analytical Methods](#). We will find such techniques to be useful in many areas of physics.

**Note:****PhET Explorations: Ladybug Motion 2D**

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

<https://archive.cnx.org/specials/317a2b1e-2fbd-11e5-99b5-e38ffb545fe6/ladybug-motion/#sim-ladybug-motion>

## Summary

- The shortest path between any two points is a straight line. In two dimensions, this path can be represented by a vector with horizontal and vertical components.
- The horizontal and vertical components of a vector are independent of one another. Motion in the horizontal direction does not affect motion

in the vertical direction, and vice versa.

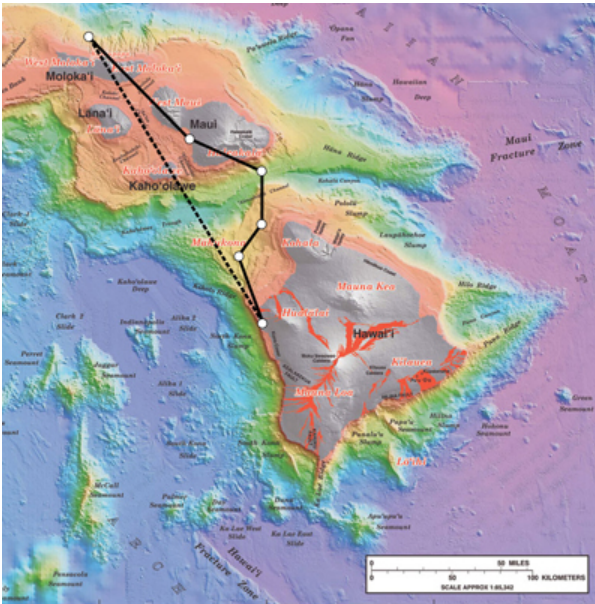
## **Glossary**

### **vector**

a quantity that has both magnitude and direction; an arrow used to represent quantities with both magnitude and direction

## Vector Addition and Subtraction: Graphical Methods

- Understand the rules of vector addition, subtraction, and multiplication.
- Apply graphical methods of vector addition and subtraction to determine the displacement of moving objects.



Displacement can be determined graphically using a scale map, such as this one of the Hawaiian Islands. A journey from Hawai'i to Moloka'i has a number of legs, or journey segments. These segments can be added graphically with a ruler to determine the total two-dimensional displacement of the journey. (credit: US Geological Survey)

## Vectors in Two Dimensions

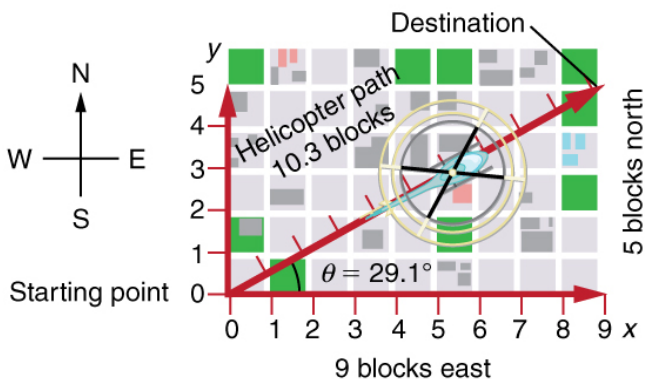
A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction of a vector relative to some reference frame (i.e., coordinate system), using an arrow having length proportional to the vector's magnitude and pointing in the direction of the vector.

[\[link\]](#) shows such a *graphical representation of a vector*, using as an example the total displacement for the person walking in a city considered in [Kinematics in Two Dimensions: An Introduction](#). We shall use the notation that a boldface symbol, such as  $\mathbf{D}$ , stands for a vector. Its magnitude is represented by the symbol in italics,  $D$ , and its direction by  $\theta$ .

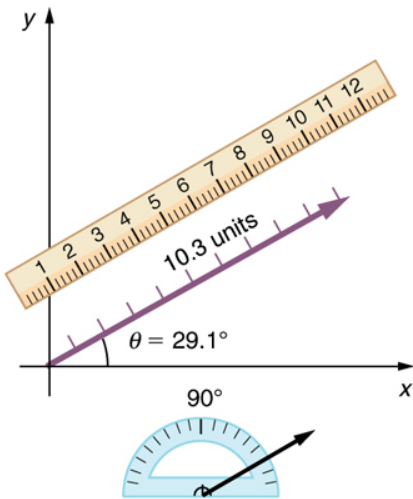
### Note:

#### Vectors in this Text

In this text, we will represent a vector with a boldface variable. For example, we will represent the quantity force with the vector  $\mathbf{F}$ , which has both magnitude and direction. The magnitude of the vector will be represented by a variable in italics, such as  $F$ , and the direction of the variable will be given by an angle  $\theta$ .



A person walks 9 blocks east and 5 blocks north. The displacement is 10.3 blocks at an angle  $29.1^\circ$  north of east.



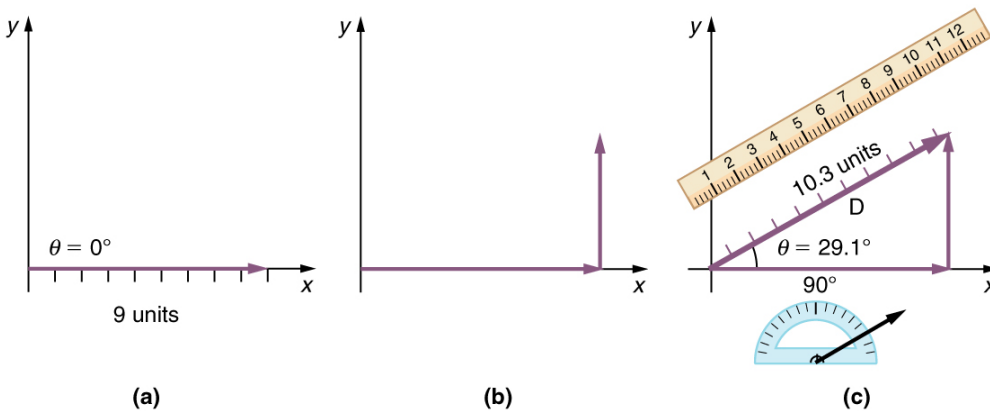
To describe the resultant vector for the person walking in a city considered in [\[link\]](#) graphically, draw an arrow to represent the total displacement vector

D. Using a protractor, draw a line at an angle  $\theta$  relative to the east-west axis. The length  $D$  of the arrow is proportional to the vector's

magnitude and is measured along the line with a ruler. In this example, the magnitude  $D$  of the vector is 10.3 units, and the direction  $\theta$  is  $29.1^\circ$  north of east.

## Vector Addition: Head-to-Tail Method

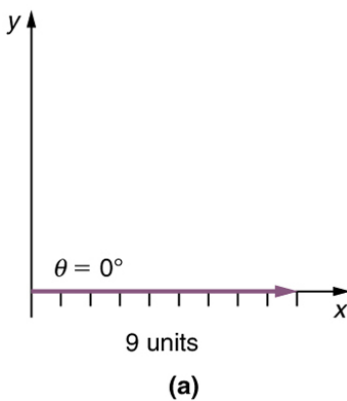
The **head-to-tail method** is a graphical way to add vectors, described in [\[link\]](#) below and in the steps following. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the final, pointed end of the arrow.



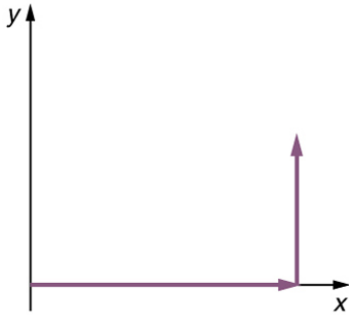
**Head-to-Tail Method:** The head-to-tail method of graphically adding vectors is illustrated for the two displacements of the person walking in a city considered in [\[link\]](#). (a) Draw a vector representing the displacement to the east. (b) Draw a vector representing the displacement to the north. The tail of this vector should originate from the head of the first, east-pointing vector.

(c) Draw a line from the tail of the east-pointing vector to the head of the north-pointing vector to form the sum or **resultant vector D**. The length of the arrow D is proportional to the vector's magnitude and is measured to be 10.3 units . Its direction, described as the angle with respect to the east (or horizontal axis)  $\theta$  is measured with a protractor to be  $29.1^\circ$ .

**Step 1.** Draw an arrow to represent the first vector (9 blocks to the east) using a ruler and protractor.



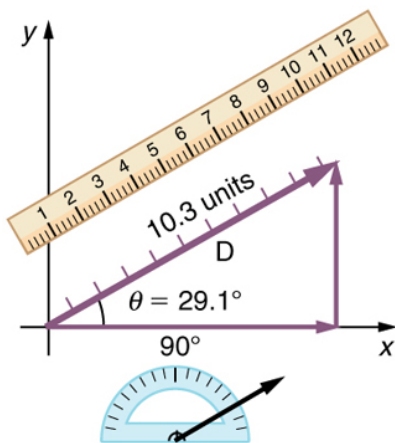
**Step 2.** Now draw an arrow to represent the second vector (5 blocks to the north). Place the tail of the second vector at the head of the first vector.



(b)

**Step 3.** If there are more than two vectors, continue this process for each vector to be added. Note that in our example, we have only two vectors, so we have finished placing arrows tip to tail.

**Step 4.** Draw an arrow from the tail of the first vector to the head of the last vector. This is the **resultant**, or the sum, of the other vectors.



(c)

**Step 5.** To get the **magnitude** of the resultant, measure its length with a ruler. (Note that in most calculations, we will use the Pythagorean theorem to determine this length.)

**Step 6.** To get the **direction** of the resultant, measure the angle it makes with the reference frame using a protractor. (Note that in most calculations, we will use trigonometric relationships to determine this angle.)



The graphical addition of vectors is limited in accuracy only by the precision with which the drawings can be made and the precision of the measuring tools. It is valid for any number of vectors.

**Example:**

**Adding Vectors Graphically Using the Head-to-Tail Method: A Woman Takes a Walk**

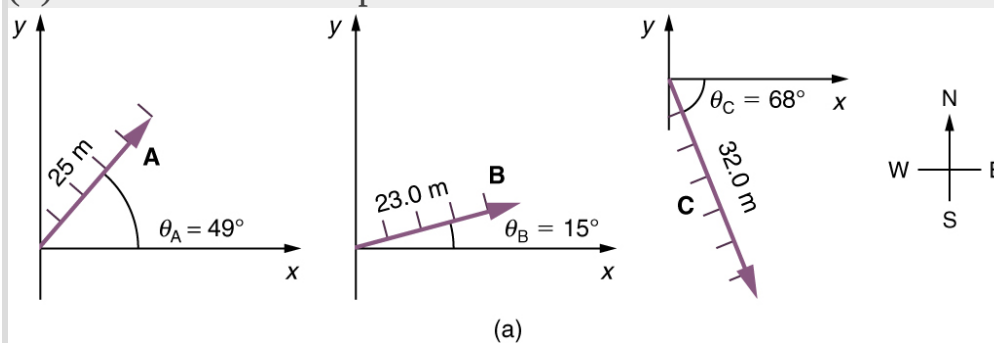
Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, she walks 25.0 m in a direction  $49.0^\circ$  north of east. Then, she walks 23.0 m heading  $15.0^\circ$  north of east. Finally, she turns and walks 32.0 m in a direction  $68.0^\circ$  south of east.

**Strategy**

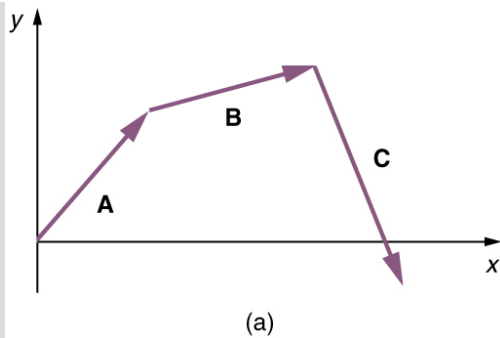
Represent each displacement vector graphically with an arrow, labeling the first **A**, the second **B**, and the third **C**, making the lengths proportional to the distance and the directions as specified relative to an east-west line. The head-to-tail method outlined above will give a way to determine the magnitude and direction of the resultant displacement, denoted **R**.

**Solution**

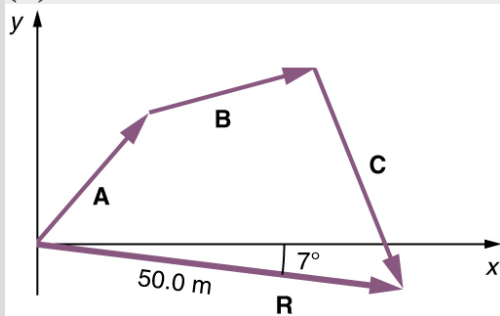
(1) Draw the three displacement vectors.



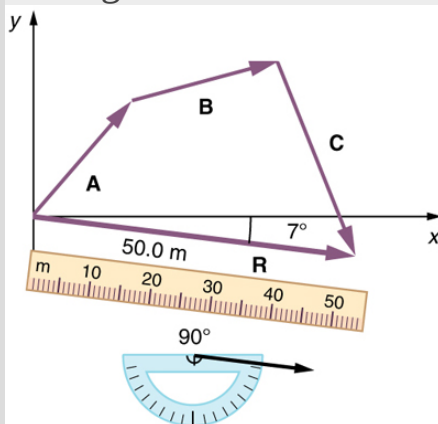
(2) Place the vectors head to tail retaining both their initial magnitude and direction.



(3) Draw the resultant vector,  $\mathbf{R}$ .



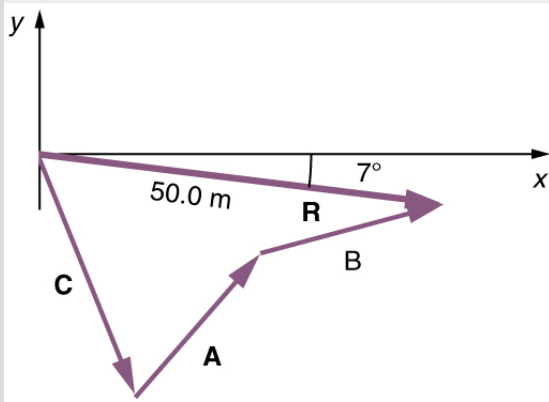
(4) Use a ruler to measure the magnitude of  $\mathbf{R}$ , and a protractor to measure the direction of  $\mathbf{R}$ . While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest horizontal or vertical axis. Since the resultant vector is south of the eastward pointing axis, we flip the protractor upside down and measure the angle between the eastward axis and the vector.



In this case, the total displacement  $\mathbf{R}$  is seen to have a magnitude of 50.0 m and to lie in a direction  $7.0^\circ$  south of east. By using its magnitude and direction, this vector can be expressed as  $R = 50.0 \text{ m}$  and  $\theta = 7.0^\circ$  south of east.

### Discussion

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that the resultant is independent of the order in which the vectors are added. Therefore, we could add the vectors in any order as illustrated in [\[link\]](#) and we will still get the same solution.



Here, we see that when the same vectors are added in a different order, the result is the same. This characteristic is true in every case and is an important characteristic of vectors. Vector addition is **commutative**. Vectors can be added in any order.

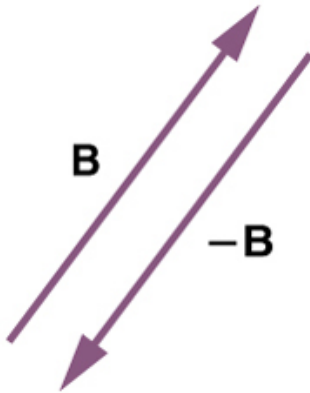
**Equation:**

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

(This is true for the addition of ordinary numbers as well—you get the same result whether you add  $2 + 3$  or  $3 + 2$ , for example).

## Vector Subtraction

Vector subtraction is a straightforward extension of vector addition. To define subtraction (say we want to subtract  $\mathbf{B}$  from  $\mathbf{A}$ , written  $\mathbf{A} - \mathbf{B}$ , we must first define what we mean by subtraction. The *negative* of a vector  $\mathbf{B}$  is defined to be  $-\mathbf{B}$ ; that is, graphically *the negative of any vector has the same magnitude but the opposite direction*, as shown in [\[link\]](#). In other words,  $\mathbf{B}$  has the same length as  $-\mathbf{B}$ , but points in the opposite direction. Essentially, we just flip the vector so it points in the opposite direction.



The negative of a vector is just another vector of the same magnitude but pointing in the opposite direction. So  $\mathbf{B}$  is the negative of  $-\mathbf{B}$ ; it has the same length but opposite direction.

The *subtraction* of vector  $\mathbf{B}$  from vector  $\mathbf{A}$  is then simply defined to be the addition of  $-\mathbf{B}$  to  $\mathbf{A}$ . Note that vector subtraction is the addition of a negative vector. The order of subtraction does not affect the results.

**Equation:**

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

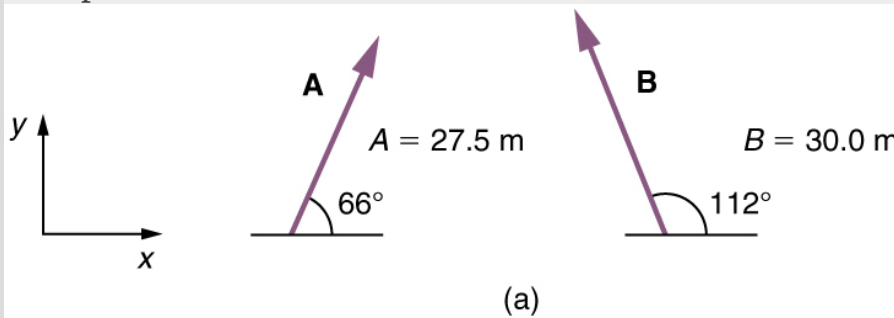
This is analogous to the subtraction of scalars (where, for example,  $5 - 2 = 5 + (-2)$ ). Again, the result is independent of the order in which

the subtraction is made. When vectors are subtracted graphically, the techniques outlined above are used, as the following example illustrates.

**Example:**

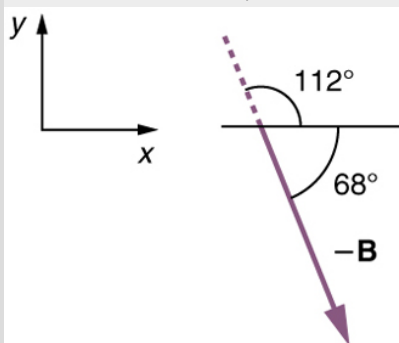
**Subtracting Vectors Graphically: A Woman Sailing a Boat**

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction  $66.0^\circ$  north of east from her current location, and then travel 30.0 m in a direction  $112^\circ$  north of east (or  $22.0^\circ$  west of north). If the woman makes a mistake and travels in the *opposite* direction for the second leg of the trip, where will she end up? Compare this location with the location of the dock.



**Strategy**

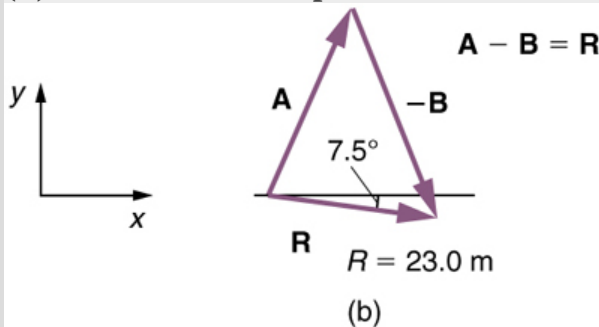
We can represent the first leg of the trip with a vector **A**, and the second leg of the trip with a vector **B**. The dock is located at a location  $\mathbf{A} + \mathbf{B}$ . If the woman mistakenly travels in the *opposite* direction for the second leg of the journey, she will travel a distance  $B$  (30.0 m) in the direction  $180^\circ - 112^\circ = 68^\circ$  south of east. We represent this as  $-\mathbf{B}$ , as shown below. The vector  $-\mathbf{B}$  has the same magnitude as **B** but is in the opposite direction. Thus, she will end up at a location  $\mathbf{A} + (-\mathbf{B})$ , or  $\mathbf{A} - \mathbf{B}$ .



We will perform vector addition to compare the location of the dock,  $\mathbf{A} + \mathbf{B}$ , with the location at which the woman mistakenly arrives,  $\mathbf{A} + (-\mathbf{B})$ .

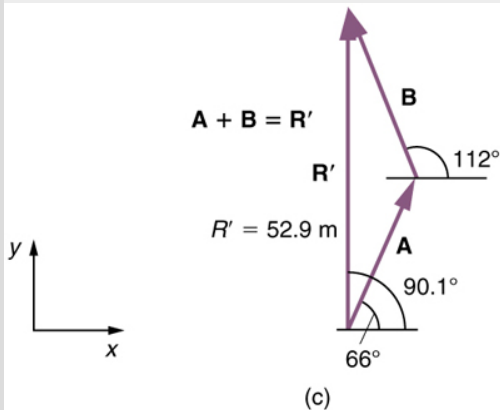
### Solution

- (1) To determine the location at which the woman arrives by accident, draw vectors  $\mathbf{A}$  and  $-\mathbf{B}$ .
- (2) Place the vectors head to tail.
- (3) Draw the resultant vector  $\mathbf{R}$ .
- (4) Use a ruler and protractor to measure the magnitude and direction of  $\mathbf{R}$ .



In this case,  $R = 23.0 \text{ m}$  and  $\theta = 7.5^\circ$  south of east.

- (5) To determine the location of the dock, we repeat this method to add vectors  $\mathbf{A}$  and  $\mathbf{B}$ . We obtain the resultant vector  $\mathbf{R}'$ :



In this case  $R = 52.9 \text{ m}$  and  $\theta = 90.1^\circ$  north of east.

We can see that the woman will end up a significant distance from the dock if she travels in the opposite direction for the second leg of the trip.

### Discussion

Because subtraction of a vector is the same as addition of a vector with the opposite direction, the graphical method of subtracting vectors works the same as for addition.

## Multiplication of Vectors and Scalars

If we decided to walk three times as far on the first leg of the trip considered in the preceding example, then we would walk  $3 \times 27.5$  m, or 82.5 m, in a direction  $66.0^\circ$  north of east. This is an example of multiplying a vector by a positive **scalar**. Notice that the magnitude changes, but the direction stays the same.

If the scalar is negative, then multiplying a vector by it changes the vector's magnitude and gives the new vector the *opposite* direction. For example, if you multiply by  $-2$ , the magnitude doubles but the direction changes. We can summarize these rules in the following way: When vector **A** is multiplied by a scalar  $c$ ,

- the magnitude of the vector becomes the absolute value of  $cA$ ,
- if  $c$  is positive, the direction of the vector does not change,
- if  $c$  is negative, the direction is reversed.

In our case,  $c = 3$  and  $A = 27.5$  m. Vectors are multiplied by scalars in many situations. Note that division is the inverse of multiplication. For example, dividing by 2 is the same as multiplying by the value  $(1/2)$ . The rules for multiplication of vectors by scalars are the same for division; simply treat the divisor as a scalar between 0 and 1.

## Resolving a Vector into Components

In the examples above, we have been adding vectors to determine the resultant vector. In many cases, however, we will need to do the opposite. We will need to take a single vector and find what other vectors added together produce it. In most cases, this involves determining the perpendicular **components** of a single vector, for example the  $x$ - and  $y$ -components, or the north-south and east-west components.

For example, we may know that the total displacement of a person walking in a city is 10.3 blocks in a direction  $29.0^\circ$  north of east and want to find out how many blocks east and north had to be walked. This method is called *finding the components (or parts)* of the displacement in the east and north

directions, and it is the inverse of the process followed to find the total displacement. It is one example of finding the components of a vector. There are many applications in physics where this is a useful thing to do. We will see this soon in [Projectile Motion](#), and much more when we cover **forces** in [Dynamics: Newton's Laws of Motion](#). Most of these involve finding components along perpendicular axes (such as north and east), so that right triangles are involved. The analytical techniques presented in [Vector Addition and Subtraction: Analytical Methods](#) are ideal for finding vector components.

**Note:**

**PhET Explorations: Maze Game**

Learn about position, velocity, and acceleration in the "Arena of Pain". Use the green arrow to move the ball. Add more walls to the arena to make the game more difficult. Try to make a goal as fast as you can.

<https://archive.cnx.org/specials/30e37034-2fbd-11e5-83a2-03be60006ece/maze-game/#sim-maze-game>

## Summary

- The **graphical method of adding vectors  $\mathbf{A}$  and  $\mathbf{B}$**  involves drawing vectors on a graph and adding them using the head-to-tail method. The resultant vector  $\mathbf{R}$  is defined such that  $\mathbf{A} + \mathbf{B} = \mathbf{R}$ . The magnitude and direction of  $\mathbf{R}$  are then determined with a ruler and protractor, respectively.
- The **graphical method of subtracting vector  $\mathbf{B}$  from  $\mathbf{A}$**  involves adding the opposite of vector  $\mathbf{B}$ , which is defined as  $-\mathbf{B}$ . In this case,  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \mathbf{R}$ . Then, the head-to-tail method of addition is followed in the usual way to obtain the resultant vector  $\mathbf{R}$ .
- Addition of vectors is **commutative** such that  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ .
- The **head-to-tail method** of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at



the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.

- If a vector  $\mathbf{A}$  is multiplied by a scalar quantity  $c$ , the magnitude of the product is given by  $cA$ . If  $c$  is positive, the direction of the product points in the same direction as  $\mathbf{A}$ ; if  $c$  is negative, the direction of the product points in the opposite direction as  $\mathbf{A}$ .

## Conceptual Questions

### Exercise:

#### Problem:

Which of the following is a vector: a person's height, the altitude on Mt. Everest, the age of the Earth, the boiling point of water, the cost of this book, the Earth's population, the acceleration of gravity?

### Exercise:

#### Problem:

Give a specific example of a vector, stating its magnitude, units, and direction.

### Exercise:

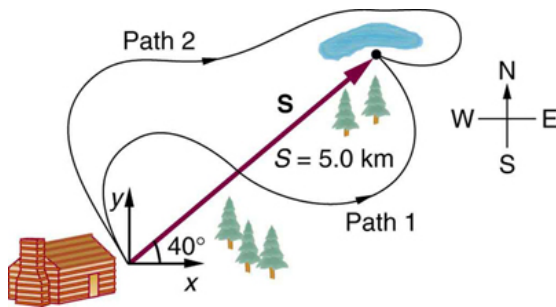
#### Problem:

What do vectors and scalars have in common? How do they differ?

### Exercise:

#### Problem:

Two campers in a national park hike from their cabin to the same spot on a lake, each taking a different path, as illustrated below. The total distance traveled along Path 1 is 7.5 km, and that along Path 2 is 8.2 km. What is the final displacement of each camper?



## Exercise:

### Problem:

If an airplane pilot is told to fly 123 km in a straight line to get from San Francisco to Sacramento, explain why he could end up anywhere on the circle shown in [\[link\]](#). What other information would he need to get to Sacramento?



## Exercise:

### Problem:

Suppose you take two steps **A** and **B** (that is, two nonzero displacements). Under what circumstances can you end up at your starting point? More generally, under what circumstances can two nonzero vectors add to give zero? Is the maximum distance you can end up from the starting point  $\mathbf{A} + \mathbf{B}$  the sum of the lengths of the two steps?

### Exercise:

**Problem:** Explain why it is not possible to add a scalar to a vector.

## Exercise:

### Problem:

If you take two steps of different sizes, can you end up at your starting point? More generally, can two vectors with different magnitudes ever add to zero? Can three or more?

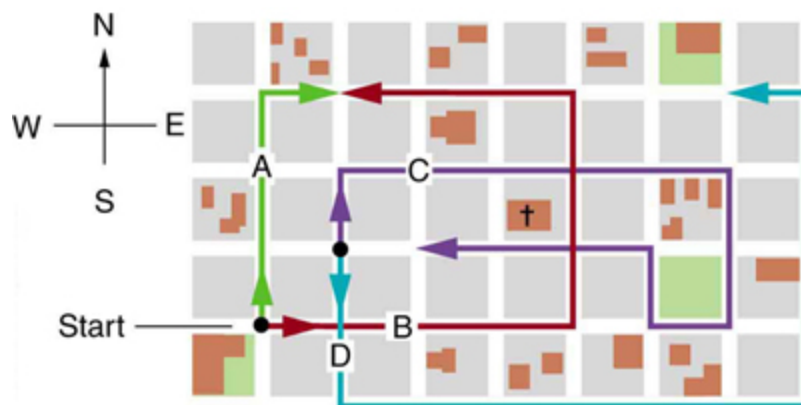
## Problems & Exercises

**Use graphical methods to solve these problems. You may assume data taken from graphs is accurate to three digits.**

### Exercise:

### Problem:

Find the following for path A in [\[link\]](#): (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.



The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

---

**Solution:**

(a) 480 m

(b) 379 m, 18.4° east of north

**Exercise:****Problem:**

Find the following for path B in [\[link\]](#): (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

**Exercise:****Problem:**

Find the north and east components of the displacement for the hikers shown in [\[link\]](#).

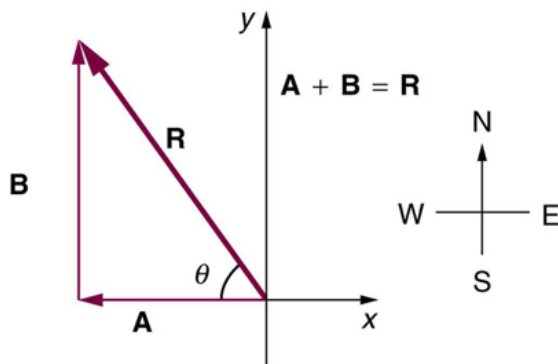
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**Solution:**

north component 3.21 km, east component 3.83 km

**Exercise:****Problem:**

Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem asks you to find their sum  $\mathbf{R} = \mathbf{A} + \mathbf{B}$ .)

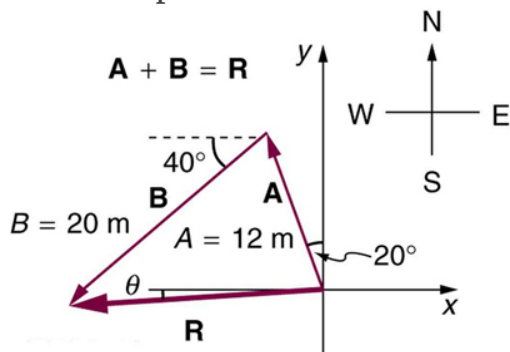


The two displacements **A** and **B** add to give a total displacement **R** having magnitude  $R$  and direction  $\theta$ .

### Exercise:

#### Problem:

Suppose you first walk 12.0 m in a direction  $20^\circ$  west of north and then 20.0 m in a direction  $40.0^\circ$  south of west. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem finds their sum  $\mathbf{R} = \mathbf{A} + \mathbf{B}$ .)



#### Solution:

19.5 m,  $4.65^\circ$  south of west

**Exercise:****Problem:**

Repeat the problem above, but reverse the order of the two legs of the walk; show that you get the same final result. That is, you first walk leg **B**, which is 20.0 m in a direction exactly  $40^\circ$  south of west, and then leg **A**, which is 12.0 m in a direction exactly  $20^\circ$  west of north. (This problem shows that  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ .)

**Exercise:****Problem:**

(a) Repeat the problem two problems prior, but for the second leg you walk 20.0 m in a direction  $40.0^\circ$  north of east (which is equivalent to subtracting **B** from **A**—that is, to finding  $\mathbf{R} = \mathbf{A} - \mathbf{B}$ ). (b) Repeat the problem two problems prior, but now you first walk 20.0 m in a direction  $40.0^\circ$  south of west and then 12.0 m in a direction  $20.0^\circ$  east of south (which is equivalent to subtracting **A** from **B**—that is, to finding  $\mathbf{R} = \mathbf{B} - \mathbf{A} = -\mathbf{R}'$ ). Show that this is the case.

---

**Solution:**

(a) 26.6 m,  $65.1^\circ$  north of east

(b) 26.6 m,  $65.1^\circ$  south of west

**Exercise:****Problem:**

Show that the *order* of addition of three vectors does not affect their sum. Show this property by choosing any three vectors **A**, **B**, and **C**, all having different lengths and directions. Find the sum  $\mathbf{A} + \mathbf{B} + \mathbf{C}$  then find their sum when added in a different order and show the result is the same. (There are five other orders in which **A**, **B**, and **C** can be added; choose only one.)

**Exercise:**

**Problem:**

Show that the sum of the vectors discussed in [\[link\]](#) gives the result shown in [\[link\]](#).

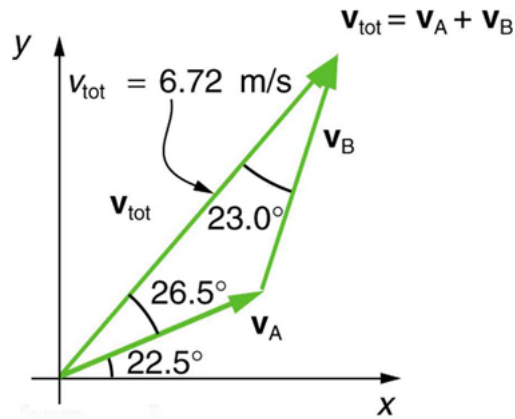
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**Solution:**

52.9 m,  $90.1^\circ$  with respect to the  $x$ -axis.

**Exercise:**

**Problem:** Find the magnitudes of velocities  $v_A$  and  $v_B$  in [\[link\]](#)



The two velocities  $\mathbf{v}_A$   
and  $\mathbf{v}_B$  add to give a total  
 $\mathbf{v}_{\text{tot}}$ .

**Exercise:****Problem:**

Find the components of  $v_{\text{tot}}$  along the  $x$ - and  $y$ -axes in [\[link\]](#).

---

**Solution:**

$x$ -component 4.41 m/s

y-component 5.07 m/s

**Exercise:**

**Problem:**

Find the components of  $v_{\text{tot}}$  along a set of perpendicular axes rotated  $30^\circ$  counterclockwise relative to those in [\[link\]](#).

## Glossary

component (of a 2-d vector)

a piece of a vector that points in either the vertical or the horizontal direction; every 2-d vector can be expressed as a sum of two vertical and horizontal vector components

commutative

refers to the interchangeability of order in a function; vector addition is commutative because the order in which vectors are added together does not affect the final sum

direction (of a vector)

the orientation of a vector in space

head (of a vector)

the end point of a vector; the location of the tip of the vector's arrowhead; also referred to as the "tip"

head-to-tail method

a method of adding vectors in which the tail of each vector is placed at the head of the previous vector

magnitude (of a vector)

the length or size of a vector; magnitude is a scalar quantity

resultant

the sum of two or more vectors



resultant vector

the vector sum of two or more vectors

scalar

a quantity with magnitude but no direction

tail

the start point of a vector; opposite to the head or tip of the arrow

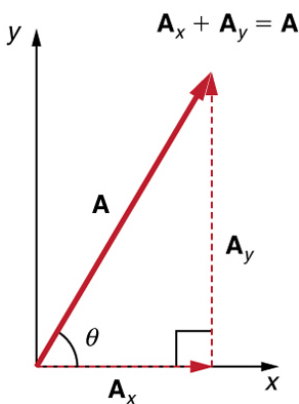
## Vector Addition and Subtraction: Analytical Methods

- Understand the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

**Analytical methods** of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

## Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like **A** in [\[link\]](#), we may wish to find which two perpendicular vectors, **A<sub>x</sub>** and **A<sub>y</sub>**, add to produce it.



The vector  $\mathbf{A}$ , with its tail at the origin of an  $x, y$ -coordinate system, is shown together with its  $x$ - and  $y$ -components,  $\mathbf{A}_x$  and  $\mathbf{A}_y$ . These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

$\mathbf{A}_x$  and  $\mathbf{A}_y$  are defined to be the components of  $\mathbf{A}$  along the  $x$ - and  $y$ -axes. The three vectors  $\mathbf{A}$ ,  $\mathbf{A}_x$ , and  $\mathbf{A}_y$  form a right triangle:

**Equation:**

$$\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}.$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if  $\mathbf{A}_x = 3$  m east,  $\mathbf{A}_y = 4$  m north, and  $\mathbf{A} = 5$  m north-east, then it is true that the vectors  $\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}$ . However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

**Equation:**

$$3 \text{ m} + 4 \text{ m} \neq 5 \text{ m}$$

Thus,

**Equation:**

$$A_x + A_y \neq A$$

If the vector **A** is known, then its magnitude  $A$  (its length) and its angle  $\theta$  (its direction) are known. To find  $A_x$  and  $A_y$ , its  $x$ - and  $y$ -components, we use the following relationships for a right triangle.

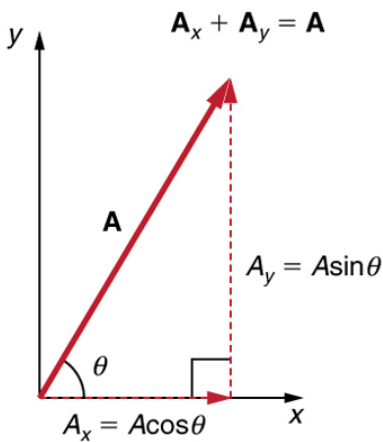
**Equation:**

$$A_x = A \cos \theta$$

and

**Equation:**

$$A_y = A \sin \theta.$$



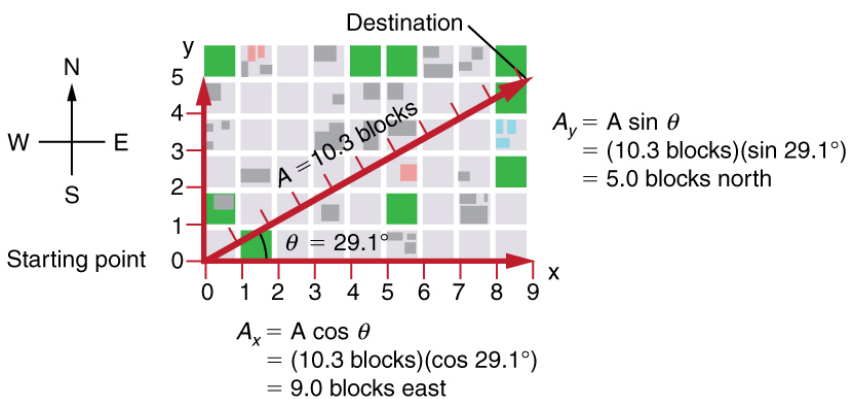
The magnitudes of  
the vector

components  $A_x$   
and  $A_y$  can be  
related to the  
resultant vector  $A$   
and the angle  $\theta$   
with trigonometric  
identities. Here we  
see that

$$A_x = A \cos \theta \text{ and}$$

$$A_y = A \sin \theta.$$

Suppose, for example, that  $A$  is the vector representing the total displacement of the person walking in a city considered in [Kinematics in Two Dimensions: An Introduction](#) and [Vector Addition and Subtraction: Graphical Methods](#).



We can use the relationships  $A_x = A \cos \theta$  and  $A_y = A \sin \theta$  to determine the magnitude of the horizontal and vertical component vectors in this example.

Then  $A = 10.3$  blocks and  $\theta = 29.1^\circ$ , so that

**Equation:**

$$A_x = A \cos \theta = (10.3 \text{ blocks})(\cos 29.1^\circ) = 9.0 \text{ blocks}$$

**Equation:**

$$A_y = A \sin \theta = (10.3 \text{ blocks})(\sin 29.1^\circ) = 5.0 \text{ blocks.}$$

## Calculating a Resultant Vector

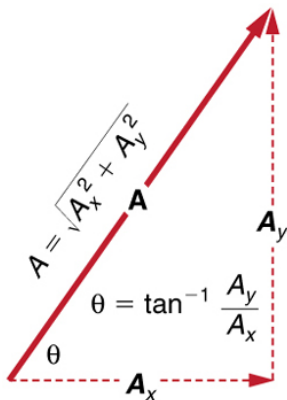
If the perpendicular components  $\mathbf{A}_x$  and  $\mathbf{A}_y$  of a vector  $\mathbf{A}$  are known, then  $\mathbf{A}$  can also be found analytically. To find the magnitude  $A$  and direction  $\theta$  of a vector from its perpendicular components  $\mathbf{A}_x$  and  $\mathbf{A}_y$ , we use the following relationships:

**Equation:**

$$A = \sqrt{A_x^2 + A_y^2}$$

**Equation:**

$$\theta = \tan^{-1}(A_y/A_x).$$



The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components  $A_x$  and  $A_y$  have been determined.

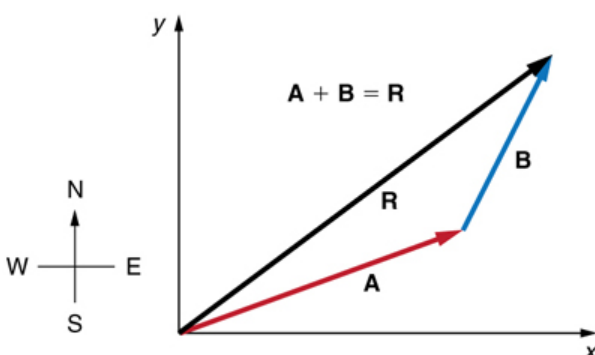
Note that the equation  $A = \sqrt{A_x^2 + A_y^2}$  is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if  $A_x$  and  $A_y$  are 9 and 5 blocks, respectively, then  $A = \sqrt{9^2 + 5^2} = 10.3$  blocks, again consistent with the example of the person walking in a city. Finally, the direction is  $\theta = \tan^{-1}(5/9) = 29.1^\circ$ , as before.

**Note:**

**Determining Vectors and Vector Components with Analytical Methods**  
Equations  $A_x = A \cos \theta$  and  $A_y = A \sin \theta$  are used to find the perpendicular components of a vector—that is, to go from  $A$  and  $\theta$  to  $A_x$  and  $A_y$ . Equations  $A = \sqrt{A_x^2 + A_y^2}$  and  $\theta = \tan^{-1}(A_y/A_x)$  are used to find a vector from its perpendicular components—that is, to go from  $A_x$  and  $A_y$  to  $A$  and  $\theta$ . Both processes are crucial to analytical methods of vector addition and subtraction.

## Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider [\[link\]](#), in which the vectors **A** and **B** are added to produce the resultant **R**.



Vectors **A** and **B** are two legs of a walk, and **R** is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of **R**.

If **A** and **B** represent two legs of a walk (two displacements), then **R** is the total displacement. The person taking the walk ends up at the tip of **R**.

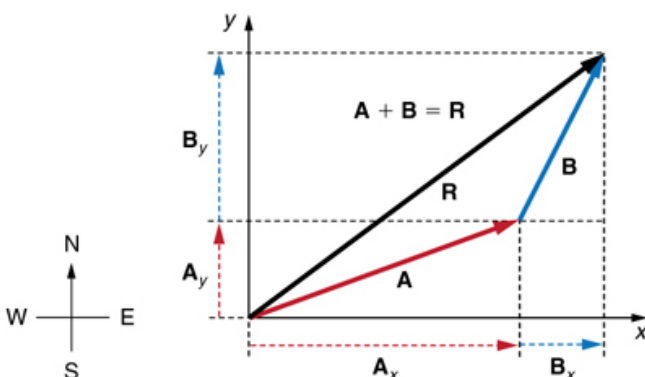
There are many ways to arrive at the same point. In particular, the person could have walked first in the *x*-direction and then in the *y*-direction. Those paths are the *x*- and *y*-components of the resultant,  $\mathbf{R}_x$  and  $\mathbf{R}_y$ . If we know

$\mathbf{R}_x$  and  $\mathbf{R}_y$ , we can find  $R$  and  $\theta$  using the equations  $A = \sqrt{A_x^2 + A_y^2}$  and  $\theta = \tan^{-1}(A_y/A_x)$ . When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

**Step 1.** Identify the *x*- and *y*-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen



perpendicular axes. Use the equations  $A_x = A \cos \theta$  and  $A_y = A \sin \theta$  to find the components. In [\[link\]](#), these components are  $A_x$ ,  $A_y$ ,  $B_x$ , and  $B_y$ . The angles that vectors **A** and **B** make with the  $x$ -axis are  $\theta_A$  and  $\theta_B$ , respectively.



To add vectors **A** and **B**, first determine the horizontal and vertical components of each vector. These are the dotted vectors  $A_x$ ,  $A_y$ ,  $B_x$  and  $B_y$  shown in the image.

**Step 2.** Find the components of the resultant along each axis by adding the components of the individual vectors along that axis. That is, as shown in [\[link\]](#),

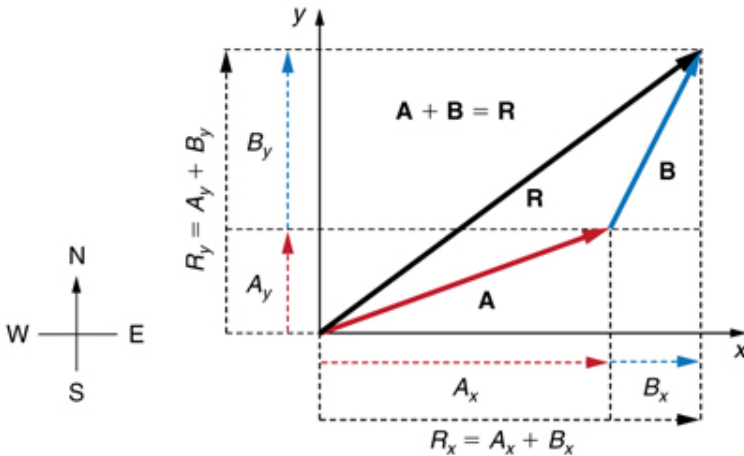
**Equation:**

$$R_x = A_x + B_x$$

and

**Equation:**

$$R_y = A_y + B_y.$$



The magnitude of the vectors  $\mathbf{A}_x$  and  $\mathbf{B}_x$  add to give the magnitude  $R_x$  of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors  $\mathbf{A}_y$  and  $\mathbf{B}_y$  add to give the magnitude  $R_y$  of the resultant vector in the vertical direction.

Components along the same axis, say the x-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the y-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of  $\mathbf{R}$  are known, its magnitude and direction can be found.

**Step 3.** To get the magnitude  $R$  of the resultant, use the Pythagorean theorem:

**Equation:**

$$R = \sqrt{R_x^2 + R_y^2}.$$

**Step 4.** To get the direction of the resultant:

**Equation:**

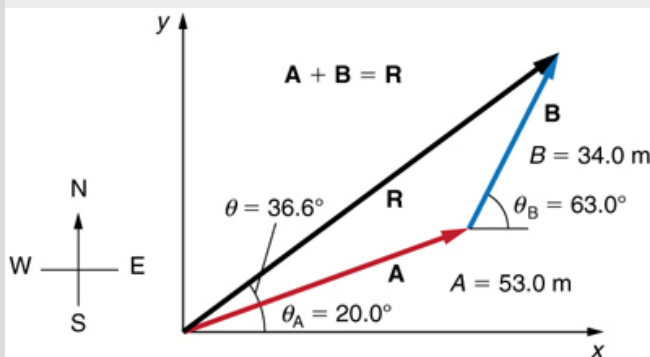
$$\theta = \tan^{-1}(R_y/R_x).$$

The following example illustrates this technique for adding vectors using perpendicular components.

**Example:**

### **Adding Vectors Using Analytical Methods**

Add the vector **A** to the vector **B** shown in [\[link\]](#), using perpendicular components along the  $x$ - and  $y$ -axes. The  $x$ - and  $y$ -axes are along the east–west and north–south directions, respectively. Vector **A** represents the first leg of a walk in which a person walks 53.0 m in a direction  $20.0^\circ$  north of east. Vector **B** represents the second leg, a displacement of 34.0 m in a direction  $63.0^\circ$  north of east.



Vector **A** has magnitude 53.0 m and direction  $20.0^\circ$  north of the  $x$ -axis. Vector **B** has magnitude 34.0 m and direction  $63.0^\circ$  north of the  $x$ -axis. You can use analytical methods to determine the magnitude and direction of **R**.

**Strategy**

The components of **A** and **B** along the  $x$ - and  $y$ -axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

**Solution**

Following the method outlined above, we first find the components of **A** and **B** along the  $x$ - and  $y$ -axes. Note that  $A = 53.0$  m,  $\theta_A = 20.0^\circ$ ,  $B = 34.0$  m, and  $\theta_B = 63.0^\circ$ . We find the  $x$ -components by using  $A_x = A \cos \theta$ , which gives

**Equation:**

$$\begin{aligned} A_x &= A \cos \theta_A = (53.0 \text{ m})(\cos 20.0^\circ) \\ &= (53.0 \text{ m})(0.940) = 49.8 \text{ m} \end{aligned}$$

and

**Equation:**

$$\begin{aligned} B_x &= B \cos \theta_B = (34.0 \text{ m})(\cos 63.0^\circ) \\ &= (34.0 \text{ m})(0.454) = 15.4 \text{ m}. \end{aligned}$$

Similarly, the  $y$ -components are found using  $A_y = A \sin \theta_A$ :

**Equation:**

$$\begin{aligned} A_y &= A \sin \theta_A = (53.0 \text{ m})(\sin 20.0^\circ) \\ &= (53.0 \text{ m})(0.342) = 18.1 \text{ m} \end{aligned}$$

and

**Equation:**

$$\begin{aligned} B_y &= B \sin \theta_B = (34.0 \text{ m})(\sin 63.0^\circ) \\ &= (34.0 \text{ m})(0.891) = 30.3 \text{ m}. \end{aligned}$$

The  $x$ - and  $y$ -components of the resultant are thus

**Equation:**

$$R_x = A_x + B_x = 49.8 \text{ m} + 15.4 \text{ m} = 65.2 \text{ m}$$

and

**Equation:**

$$R_y = A_y + B_y = 18.1 \text{ m} + 30.3 \text{ m} = 48.4 \text{ m}.$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

**Equation:**

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2} \text{ m}$$

so that

**Equation:**

$$R = 81.2 \text{ m}.$$

Finally, we find the direction of the resultant:

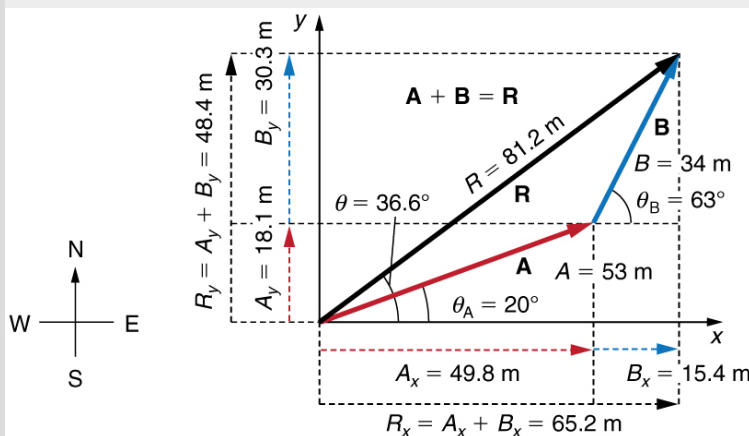
**Equation:**

$$\theta = \tan^{-1}(R_y/R_x) = +\tan^{-1}(48.4/65.2).$$

Thus,

**Equation:**

$$\theta = \tan^{-1}(0.742) = 36.6^\circ.$$



Using analytical methods, we see that the magnitude of  $\mathbf{R}$  is 81.2 m and its

direction is  $36.6^\circ$  north of east.

### Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector.

That is,  $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B})$ . Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition.*

The components of  $-\mathbf{B}$  are the negatives of the components of  $\mathbf{B}$ . The x- and y-components of the resultant  $\mathbf{A} - \mathbf{B} = \mathbf{R}$  are thus

**Equation:**

$$R_x = A_x + (-B_x)$$

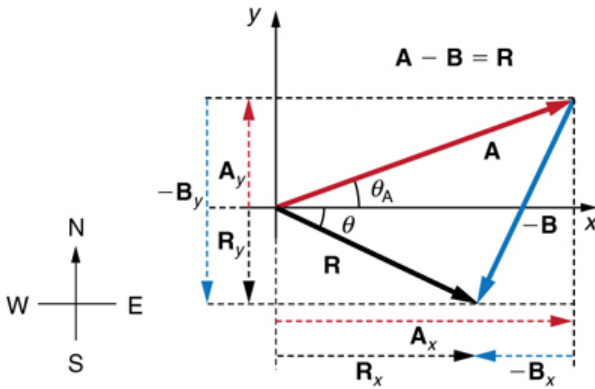
and

**Equation:**

$$R_y = A_y + (-B_y)$$

and the rest of the method outlined above is identical to that for addition. (See [\[link\]](#).)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, [Projectile Motion](#), is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.



The subtraction of the two vectors shown in [\[link\]](#). The components of  $-\mathbf{B}$  are the negatives of the components of  $\mathbf{B}$ . The method of subtraction is the same as that for addition.

### Note:

#### PhET Explorations: Vector Addition

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

[https://phet.colorado.edu/sims/vector-addition/vector-addition\\_en.html](https://phet.colorado.edu/sims/vector-addition/vector-addition_en.html)

## Summary

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors  $\mathbf{A}$  and  $\mathbf{B}$  using the analytical method are as follows:

Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

**Equation:**

$$\begin{aligned}A_x &= A \cos \theta \\B_x &= B \cos \theta\end{aligned}$$

and

**Equation:**

$$\begin{aligned}A_y &= A \sin \theta \\B_y &= B \sin \theta.\end{aligned}$$

Step 2: Add the horizontal and vertical components of each vector to determine the components  $R_x$  and  $R_y$  of the resultant vector, **R**:

**Equation:**

$$R_x = A_x + B_x$$

and

**Equation:**

$$R_y = A_y + B_y.$$

Step 3: Use the Pythagorean theorem to determine the magnitude,  $R$ , of the resultant vector **R**:

**Equation:**

$$R = \sqrt{R_x^2 + R_y^2}.$$

Step 4: Use a trigonometric identity to determine the direction,  $\theta$ , of **R** :



**Equation:**

$$\theta = \tan^{-1}(R_y/R_x).$$

## Conceptual Questions

**Exercise:**

**Problem:**

Suppose you add two vectors **A** and **B**. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

**Exercise:**

**Problem:**

Give an example of a nonzero vector that has a component of zero.

**Exercise:**

**Problem:**

Explain why a vector cannot have a component greater than its own magnitude.

**Exercise:**

**Problem:**

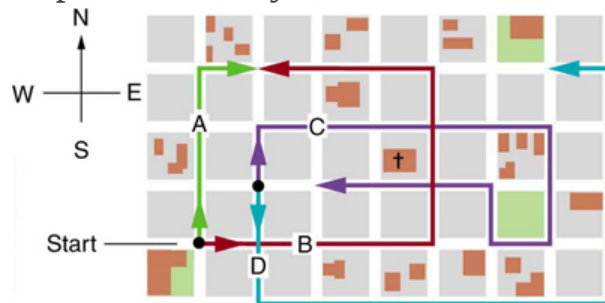
If the vectors **A** and **B** are perpendicular, what is the component of **A** along the direction of **B**? What is the component of **B** along the direction of **A**?

## Problems & Exercises

**Exercise:**

**Problem:**

Find the following for path C in [\[link\]](#): (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.



The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

---

**Solution:**

(a) 1.56 km

(b) 120 m east

**Exercise:****Problem:**

Find the following for path D in [\[link\]](#): (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

**Exercise:**

**Problem:**

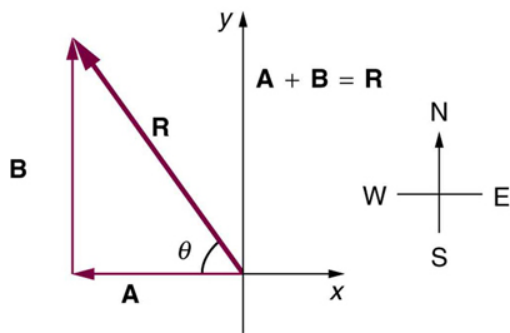
Find the north and east components of the displacement from San Francisco to Sacramento shown in [\[link\]](#).

**Solution:**

North-component 87.0 km, east-component 87.0 km

**Exercise:****Problem:**

Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem asks you to find their sum  $\mathbf{R} = \mathbf{A} + \mathbf{B}$ .)



The two displacements **A** and **B** add to give a total displacement **R** having magnitude  $R$  and direction  $\theta$ .

Note that you can also solve this graphically. Discuss why the analytical technique for solving this problem is potentially more accurate than the graphical technique.

### Exercise:

#### Problem:

Repeat [\[link\]](#) using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is,  $\mathbf{B} + \mathbf{A} = \mathbf{A} + \mathbf{B}$ .) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking your other path.

---

#### Solution:

30.8 m, 35.8 west of north

### Exercise:

**Problem:**

You drive 7.50 km in a straight line in a direction  $15^\circ$  east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to find the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

**Exercise:****Problem:**

Do [\[link\]](#) again using analytical techniques and change the second leg of the walk to 25.0 m straight south. (This is equivalent to subtracting  $\mathbf{B}$  from  $\mathbf{A}$ —that is, finding  $\mathbf{R} = \mathbf{A} - \mathbf{B}$ ) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract  $\mathbf{A}$  from  $\mathbf{B}$ —that is, to find  $\mathbf{A} = \mathbf{B} + \mathbf{C}$ . Is that consistent with your result?)

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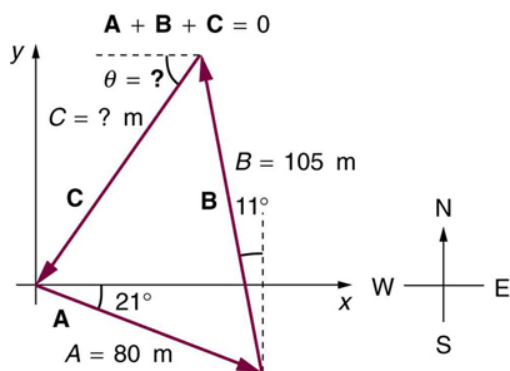
**Solution:**

(a) 30.8 m,  $54.2^\circ$  south of west

(b) 30.8 m,  $54.2^\circ$  north of east

**Exercise:****Problem:**

A new landowner has a triangular piece of flat land she wishes to fence. Starting at the west corner, she measures the first side to be 80.0 m long and the next to be 105 m. These sides are represented as displacement vectors  $\mathbf{A}$  from  $\mathbf{B}$  in [\[link\]](#). She then correctly calculates the length and orientation of the third side  $\mathbf{C}$ . What is her result?



### Exercise:

#### Problem:

You fly 32.0 km in a straight line in still air in the direction  $35.0^\circ$  south of west. (a) Find the distances you would have to fly straight south and then straight west to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the south and west directions.) (b) Find the distances you would have to fly first in a direction  $45.0^\circ$  south of west and then in a direction  $45.0^\circ$  west of north. These are the components of the displacement along a different set of axes—one rotated  $45^\circ$ .

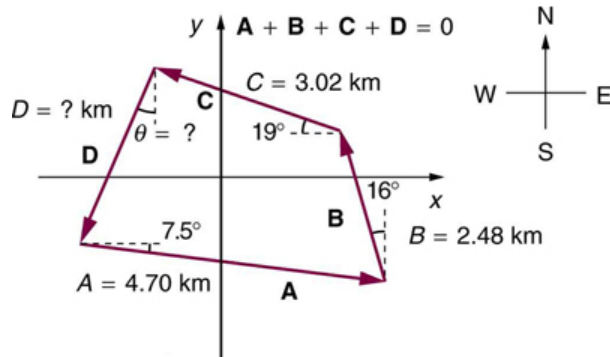
#### Solution:

18.4 km south, then 26.2 km west (b) 31.5 km at  $45.0^\circ$  south of west, then 5.56 km at  $45.0^\circ$  west of north

### Exercise:

#### Problem:

A farmer wants to fence off his four-sided plot of flat land. He measures the first three sides, shown as **A**, **B**, and **C** in [\[link\]](#), and then correctly calculates the length and orientation of the fourth side **D**. What is his result?



### Exercise:

#### Problem:

In an attempt to escape his island, Gilligan builds a raft and sets to sea. The wind shifts a great deal during the day, and he is blown along the following straight lines: 2.50 km  $45.0^\circ$  north of west; then 4.70 km  $60.0^\circ$  south of east; then 1.30 km  $25.0^\circ$  south of west; then 5.10 km straight east; then 1.70 km  $5.00^\circ$  east of north; then 7.20 km  $55.0^\circ$  south of west; and finally 2.80 km  $10.0^\circ$  north of east. What is his final position relative to the island?

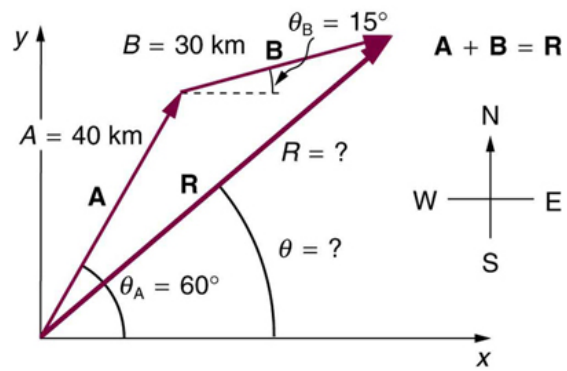
#### Solution:

7.34 km,  $63.5^\circ$  south of east

### Exercise:

#### Problem:

Suppose a pilot flies 40.0 km in a direction  $60^\circ$  north of east and then flies 30.0 km in a direction  $15^\circ$  north of east as shown in [\[link\]](#). Find her total distance  $R$  from the starting point and the direction  $\theta$  of the straight-line path to the final position. Discuss qualitatively how this flight would be altered by a wind from the north and how the effect of the wind would depend on both wind speed and the speed of the plane relative to the air mass.



## Glossary

analytical method

the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities



## Projectile Motion

- Identify and explain the properties of a projectile, such as acceleration due to gravity, range, maximum height, and trajectory.
- Determine the location and velocity of a projectile at different points in its trajectory.
- Apply the principle of independence of motion to solve projectile motion problems.

**Projectile motion** is the **motion** of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a **projectile**, and its path is called its **trajectory**. The motion of falling objects, as covered in [Problem-Solving Basics for One-Dimensional Kinematics](#), is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, such as that of a football or other object for which **air resistance is negligible**.

The most important fact to remember here is that *motions along perpendicular axes are independent* and thus can be analyzed separately. This fact was discussed in [Kinematics in Two Dimensions: An Introduction](#), where vertical and horizontal motions were seen to be independent. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible, because acceleration due to gravity is vertical—thus, there will be no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the  $x$ -axis and the vertical axis the  $y$ -axis. [\[link\]](#) illustrates the notation for displacement, where  $\mathbf{s}$  is defined to be the total displacement and  $\mathbf{x}$  and  $\mathbf{y}$  are its components along the horizontal and vertical axes, respectively. The magnitudes of these vectors are  $s$ ,  $x$ , and  $y$ . (Note that in the last section we used the notation  $\mathbf{A}$  to represent a vector with components  $\mathbf{A}_x$  and  $\mathbf{A}_y$ . If we continued this format, we would call displacement  $\mathbf{s}$  with components  $\mathbf{s}_x$  and  $\mathbf{s}_y$ . However, to simplify the notation, we will simply represent the component vectors as  $\mathbf{x}$  and  $\mathbf{y}$ .)

Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the  $x$ - and  $y$ -axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple:  $a_y = -g = -9.80 \text{ m/s}^2$ . (Note that this definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.) Because gravity is vertical,  $a_x = 0$ . Both accelerations are constant, so the kinematic equations can be used.

### Note:

Review of Kinematic Equations (constant  $a$ )

#### Equation:

$$x = x_0 + \bar{v}t$$

#### Equation:

$$\bar{v} = \frac{v_0 + v}{2}$$

#### Equation:

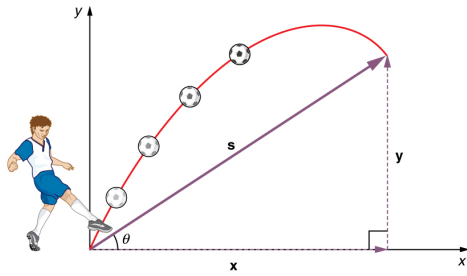
$$v = v_0 + at$$

#### Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

#### Equation:

$$v^2 = v_0^2 + 2a(x - x_0).$$



The total displacement  $\mathbf{s}$  of a soccer ball at a point along its path. The vector  $\mathbf{s}$  has components  $\mathbf{x}$  and  $\mathbf{y}$  along the horizontal and vertical axes. Its magnitude is  $s$ , and it makes an angle  $\theta$  with the horizontal.

Given these assumptions, the following steps are then used to analyze projectile motion:

**Step 1.** Resolve or break the motion into horizontal and vertical components along the  $x$ - and  $y$ -axes. These axes are perpendicular, so  $A_x = A \cos \theta$  and  $A_y = A \sin \theta$  are used. The magnitude of the components of displacement  $\mathbf{s}$  along these axes are  $x$  and  $y$ . The magnitudes of the components of the velocity  $\mathbf{v}$  are  $v_x = v \cos \theta$  and  $v_y = v \sin \theta$ , where  $v$  is the magnitude of the velocity and  $\theta$  is its direction, as shown in [\[link\]](#). Initial values are denoted with a subscript 0, as usual.

**Step 2.** Treat the motion as two independent one-dimensional motions, one horizontal and the other vertical. The kinematic equations for horizontal and vertical motion take the following forms:

**Equation:**

$$\text{Horizontal Motion}(a_x = 0)$$

**Equation:**

$$x = x_0 + v_x t$$

**Equation:**

$$v_x = v_{0x} = v_x = \text{velocity is a constant.}$$

**Equation:**

$$\text{Vertical Motion(assuming positive is up } a_y = -g = -9.80\text{m/s}^2)$$

**Equation:**

$$y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$$

**Equation:**

$$v_y = v_{0y} - gt$$

**Equation:**

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

**Equation:**

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

**Step 3.** Solve for the unknowns in the two separate motions—one horizontal and one vertical. Note that the only common variable between the motions is time  $t$ . The problem solving procedures here are the same as for one-dimensional **kinematics** and are illustrated in the solved examples below.

**Step 4.** Recombine the two motions to find the total displacement  $\mathbf{s}$  and velocity  $\mathbf{v}$ . Because the  $x$  - and  $y$  -motions are perpendicular, we determine these vectors by using the techniques outlined in the [Vector Addition and Subtraction: Analytical Methods](#) and employing  $A = \sqrt{A_x^2 + A_y^2}$  and  $\theta = \tan^{-1}(A_y/A_x)$  in the following form, where  $\theta$  is the direction of the displacement  $\mathbf{s}$  and  $\theta_v$  is the direction of the velocity  $\mathbf{v}$ :

**Total displacement and velocity**

**Equation:**

$$s = \sqrt{x^2 + y^2}$$

**Equation:**

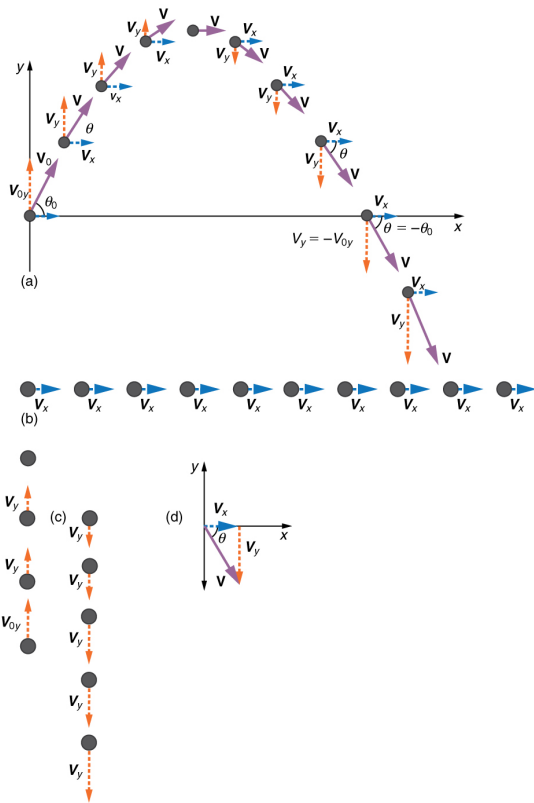
$$\theta = \tan^{-1}(y/x)$$

**Equation:**

$$v = \sqrt{v_x^2 + v_y^2}$$

**Equation:**

$$\theta_v = \tan^{-1}(v_y/v_x).$$



- (a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. (b) The horizontal motion is simple, because  $a_x = 0$  and  $v_x$  is thus constant. (c) The velocity in the vertical direction begins to decrease as the object rises; at its highest point, the vertical velocity is zero. As the object falls towards the Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. (d) The  $x$  - and  $y$  -motions are recombined to give the total velocity at any given point on the trajectory.

### Example:

#### A Fireworks Projectile Explodes High and Away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of  $75.0^\circ$  above the horizontal, as illustrated in [\[link\]](#). The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passed between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes?

#### Strategy

Because air resistance is negligible for the unexploded shell, the analysis method outlined above can be used. The motion can be broken into horizontal and vertical motions in which  $a_x = 0$  and  $a_y = -g$ . We can then define  $x_0$

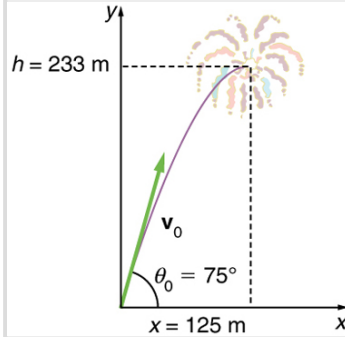
and  $y_0$  to be zero and solve for the desired quantities.

**Solution for (a)**

By “height” we mean the altitude or vertical position  $y$  above the starting point. The highest point in any trajectory, called the apex, is reached when  $v_y = 0$ . Since we know the initial and final velocities as well as the initial position, we use the following equation to find  $y$ :

**Equation:**

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$



The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Because  $y_0$  and  $v_y$  are both zero, the equation simplifies to

**Equation:**

$$0 = v_{0y}^2 - 2gy.$$

Solving for  $y$  gives

**Equation:**

$$y = \frac{v_{0y}^2}{2g}.$$

Now we must find  $v_{0y}$ , the component of the initial velocity in the  $y$ -direction. It is given by  $v_{0y} = v_0 \sin \theta$ , where  $v_0$  is the initial velocity of 70.0 m/s, and  $\theta_0 = 75.0^\circ$  is the initial angle. Thus,

**Equation:**

$$v_{0y} = v_0 \sin \theta_0 = (70.0 \text{ m/s})(\sin 75^\circ) = 67.6 \text{ m/s}.$$

and  $y$  is

**Equation:**

$$y = \frac{(67.6 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)},$$

so that

**Equation:**

$$y = 233\text{m}.$$

**Discussion for (a)**

Note that because up is positive, the initial velocity is positive, as is the maximum height, but the acceleration due to gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6 m/s initial vertical component of velocity will reach a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, and so the initial velocity would have to be somewhat larger than that given to reach the same height.

**Solution for (b)**

As in many physics problems, there is more than one way to solve for the time to the highest point. In this case, the easiest method is to use  $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$ . Because  $y_0$  is zero, this equation reduces to simply

**Equation:**

$$y = \frac{1}{2}(v_{0y} + v_y)t.$$

Note that the final vertical velocity,  $v_y$ , at the highest point is zero. Thus,

**Equation:**

$$\begin{aligned} t &= \frac{2y}{(v_{0y} + v_y)} = \frac{2(233 \text{ m})}{(67.6 \text{ m/s})} \\ &= 6.90 \text{ s}. \end{aligned}$$

**Discussion for (b)**

This time is also reasonable for large fireworks. When you are able to see the launch of fireworks, you will notice several seconds pass before the shell explodes. (Another way of finding the time is by using  $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$ , and solving the quadratic equation for  $t$ .)

**Solution for (c)**

Because air resistance is negligible,  $a_x = 0$  and the horizontal velocity is constant, as discussed above. The horizontal displacement is horizontal velocity multiplied by time as given by  $x = x_0 + v_x t$ , where  $x_0$  is equal to zero:

**Equation:**

$$x = v_x t,$$

where  $v_x$  is the x-component of the velocity, which is given by  $v_x = v_0 \cos \theta_0$ . Now,

**Equation:**

$$v_x = v_0 \cos \theta_0 = (70.0 \text{ m/s})(\cos 75.0^\circ) = 18.1 \text{ m/s}.$$

The time  $t$  for both motions is the same, and so  $x$  is

**Equation:**

$$x = (18.1 \text{ m/s})(6.90 \text{ s}) = 125 \text{ m}.$$

**Discussion for (c)**

The horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. Once the shell explodes, air resistance has a major effect, and many fragments will land directly below.

In solving part (a) of the preceding example, the expression we found for  $y$  is valid for any projectile motion where air resistance is negligible. Call the maximum height  $y = h$ ; then,

**Equation:**

$$h = \frac{v_{0y}^2}{2g}.$$

This equation defines the *maximum height of a projectile* and depends only on the vertical component of the initial velocity.

**Note:**

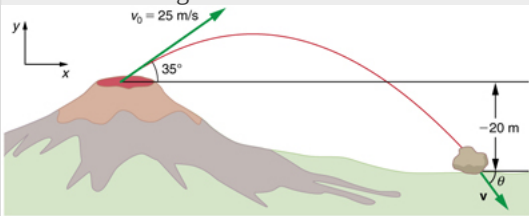
**Defining a Coordinate System**

It is important to set up a coordinate system when analyzing projectile motion. One part of defining the coordinate system is to define an origin for the  $x$  and  $y$  positions. Often, it is convenient to choose the initial position of the object as the origin such that  $x_0 = 0$  and  $y_0 = 0$ . It is also important to define the positive and negative directions in the  $x$  and  $y$  directions. Typically, we define the positive vertical direction as upwards, and the positive horizontal direction is usually the direction of the object's motion. When this is the case, the vertical acceleration,  $g$ , takes a negative value (since it is directed downwards towards the Earth). However, it is occasionally useful to define the coordinates differently. For example, if you are analyzing the motion of a ball thrown downwards from the top of a cliff, it may make sense to define the positive direction downwards since the motion of the ball is solely in the downwards direction. If this is the case,  $g$  takes a positive value.

**Example:**

**Calculating Projectile Motion: Hot Rock Projectile**

Kilauea in Hawaii is the world's most continuously active volcano. Very active volcanoes characteristically eject red-hot rocks and lava rather than smoke and ash. Suppose a large rock is ejected from the volcano with a speed of 25.0 m/s and at an angle  $35.0^\circ$  above the horizontal, as shown in [link](#). The rock strikes the side of the volcano at an altitude 20.0 m lower than its starting point. (a) Calculate the time it takes the rock to follow this path. (b) What are the magnitude and direction of the rock's velocity at impact?



The trajectory of a rock ejected from the Kilauea volcano.

**Strategy**

Again, resolving this two-dimensional motion into two independent one-dimensional motions will allow us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. We will solve for  $t$  first. While the rock is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, the vertical and horizontal results will be recombined to obtain  $v$  and  $\theta_v$  at the final time  $t$  determined in the first part of the example.

**Solution for (a)**

While the rock is in the air, it rises and then falls to a final position 20.0 m lower than its starting altitude. We can find the time for this by using

**Equation:**

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2.$$

If we take the initial position  $y_0$  to be zero, then the final position is  $y = -20.0$  m. Now the initial vertical velocity is the vertical component of the initial velocity, found from  $v_{0y} = v_0 \sin \theta_0 = (25.0 \text{ m/s})(\sin 35.0^\circ) = 14.3 \text{ m/s}$ . Substituting known values yields

**Equation:**

$$-20.0 \text{ m} = (14.3 \text{ m/s})t - (4.90 \text{ m/s}^2)t^2.$$

Rearranging terms gives a quadratic equation in  $t$ :

**Equation:**

$$(4.90 \text{ m/s}^2)t^2 - (14.3 \text{ m/s})t - (20.0 \text{ m}) = 0.$$

This expression is a quadratic equation of the form  $at^2 + bt + c = 0$ , where the constants are  $a = 4.90$ ,  $b = -14.3$ , and  $c = -20.0$ . Its solutions are given by the quadratic formula:

**Equation:**

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This equation yields two solutions:  $t = 3.96$  and  $t = -1.03$ . (It is left as an exercise for the reader to verify these solutions.) The time is  $t = 3.96$  s or  $-1.03$  s. The negative value of time implies an event before the start of motion, and so we discard it. Thus,

**Equation:**

$$t = 3.96 \text{ s}.$$

#### Discussion for (a)

The time for projectile motion is completely determined by the vertical motion. So any projectile that has an initial vertical velocity of  $14.3 \text{ m/s}$  and lands  $20.0 \text{ m}$  below its starting altitude will spend  $3.96 \text{ s}$  in the air.

#### Solution for (b)

From the information now in hand, we can find the final horizontal and vertical velocities  $v_x$  and  $v_y$  and combine them to find the total velocity  $v$  and the angle  $\theta_0$  it makes with the horizontal. Of course,  $v_x$  is constant so we can solve for it at any horizontal location. In this case, we chose the starting point since we know both the initial velocity and initial angle. Therefore:

**Equation:**

$$v_x = v_0 \cos \theta_0 = (25.0 \text{ m/s})(\cos 35^\circ) = 20.5 \text{ m/s}.$$

The final vertical velocity is given by the following equation:

**Equation:**

$$v_y = v_{0y} - gt,$$

where  $v_{0y}$  was found in part (a) to be  $14.3 \text{ m/s}$ . Thus,

**Equation:**

$$v_y = 14.3 \text{ m/s} - (9.80 \text{ m/s}^2)(3.96 \text{ s})$$

so that

**Equation:**

$$v_y = -24.5 \text{ m/s}.$$

To find the magnitude of the final velocity  $v$  we combine its perpendicular components, using the following equation:

**Equation:**



$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.5 \text{ m/s})^2 + (-24.5 \text{ m/s})^2},$$

which gives

**Equation:**

$$v = 31.9 \text{ m/s}.$$

The direction  $\theta_v$  is found from the equation:

**Equation:**

$$\theta_v = \tan^{-1}(v_y/v_x)$$

so that

**Equation:**

$$\theta_v = \tan^{-1}(-24.5/20.5) = \tan^{-1}(-1.19).$$

Thus,

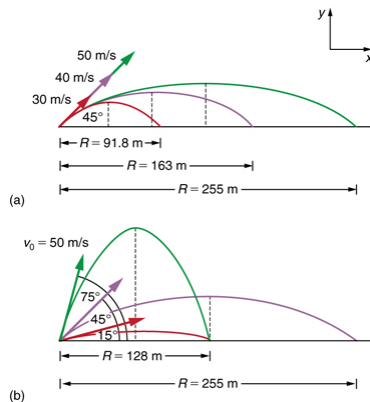
**Equation:**

$$\theta_v = -50.1^\circ.$$

#### Discussion for (b)

The negative angle means that the velocity is  $50.1^\circ$  below the horizontal. This result is consistent with the fact that the final vertical velocity is negative and hence downward—as you would expect because the final altitude is 20.0 m lower than the initial altitude. (See [\[link\]](#).)

One of the most important things illustrated by projectile motion is that vertical and horizontal motions are independent of each other. Galileo was the first person to fully comprehend this characteristic. He used it to predict the range of a projectile. On level ground, we define **range** to be the horizontal distance  $R$  traveled by a projectile. Galileo and many others were interested in the range of projectiles primarily for military purposes—such as aiming cannons. However, investigating the range of projectiles can shed light on other interesting phenomena, such as the orbits of satellites around the Earth. Let us consider projectile range further.



Trajectories of projectiles on level ground. (a) The greater the initial speed  $v_0$ , the greater the range for a given initial angle. (b) The effect of initial angle  $\theta_0$  on the range of a projectile with a

given initial speed. Note that the range is the same for  $15^\circ$  and  $75^\circ$ , although the maximum heights of those paths are different.

How does the initial velocity of a projectile affect its range? Obviously, the greater the initial speed  $v_0$ , the greater the range, as shown in [\[link\]](#)(a). The initial angle  $\theta_0$  also has a dramatic effect on the range, as illustrated in [\[link\]](#)(b). For a fixed initial speed, such as might be produced by a cannon, the maximum range is obtained with  $\theta_0 = 45^\circ$ . This is true only for conditions neglecting air resistance. If air resistance is considered, the maximum angle is approximately  $38^\circ$ . Interestingly, for every initial angle except  $45^\circ$ , there are two angles that give the same range—the sum of those angles is  $90^\circ$ . The range also depends on the value of the acceleration of gravity  $g$ . The lunar astronaut Alan Shepherd was able to drive a golf ball a great distance on the Moon because gravity is weaker there. The range  $R$  of a projectile on *level ground* for which air resistance is negligible is given by

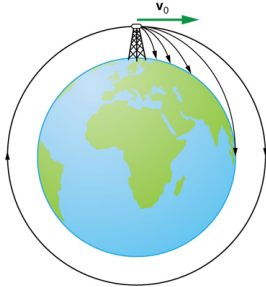
**Equation:**

$$R = \frac{v_0^2 \sin 2\theta_0}{g},$$

where  $v_0$  is the initial speed and  $\theta_0$  is the initial angle relative to the horizontal. The proof of this equation is left as an end-of-chapter problem (hints are given), but it does fit the major features of projectile range as described.

When we speak of the range of a projectile on level ground, we assume that  $R$  is very small compared with the circumference of the Earth. If, however, the range is large, the Earth curves away below the projectile and acceleration of gravity changes direction along the path. The range is larger than predicted by the range equation given above because the projectile has farther to fall than it would on level ground. (See [\[link\]](#).) If the initial speed is great enough, the projectile goes into orbit. This possibility was recognized centuries before it could be accomplished. When an object is in orbit, the Earth curves away from underneath the object at the same rate as it falls. The object thus falls continuously but never hits the surface. These and other aspects of orbital motion, such as the rotation of the Earth, will be covered analytically and in greater depth later in this text.

Once again we see that thinking about one topic, such as the range of a projectile, can lead us to others, such as the Earth orbits. In [Addition of Velocities](#), we will examine the addition of velocities, which is another important aspect of two-dimensional kinematics and will also yield insights beyond the immediate topic.



Projectile to satellite. In each case shown here, a projectile is launched from a very high tower to avoid air resistance. With increasing initial speed, the

range increases and becomes longer than it would be on level ground because the Earth curves away underneath its path. With a large enough initial speed, orbit is achieved.

**Note:**

**PhET Explorations: Projectile Motion**

Blast a Buick out of a cannon! Learn about projectile motion by firing various objects. Set the angle, initial speed, and mass. Add air resistance. Make a game out of this simulation by trying to hit a target.

[https://phet.colorado.edu/sims/projectile-motion/projectile-motion\\_en.html](https://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html)

**Summary**

- Projectile motion is the motion of an object through the air that is subject only to the acceleration of gravity.
- To solve projectile motion problems, perform the following steps:

Determine a coordinate system. Then, resolve the position and/or velocity of the object in the horizontal and vertical components. The components of position are given by the quantities  $x$  and  $y$ , and the components of the velocity  $\mathbf{v}$  are  $v_x = v \cos \theta$  and  $v_y = v \sin \theta$ , where  $v$  is the magnitude of the velocity and  $\theta$  is its direction.

Analyze the motion of the projectile in the horizontal direction using the following equations:

**Equation:**

Horizontal motion ( $a_x = 0$ )

**Equation:**

$$x = x_0 + v_x t$$

**Equation:**

$v_x = v_{0x} = \mathbf{v}_x = \text{velocity is a constant.}$

Analyze the motion of the projectile in the vertical

**Equation:**

**Equation:**

Vertical motion (Assuming positive direction is up;  $a_y = -g = -9.80 \text{ m/s}^2$ )  
 $y = y_0 + \frac{1}{2}(v_{0y})$

direction  
using the  
following  
equations:

Recombine the  
horizontal and  
vertical components  
of location and/or  
velocity using the  
following equations:

**Equation:**

$$s = \sqrt{x^2 + y^2}$$

**Equation:**

$$\theta = \tan^{-1}(y/x)$$

**Equation:**

$$v = \sqrt{v_x^2 + v_y^2}$$

**Equation:**

$$\theta_v = \tan^{-1}(v_y/v_x).$$

- The maximum height  $h$  of a projectile launched with initial vertical velocity  $v_{0y}$  is given by

**Equation:**

$$h = \frac{v_{0y}^2}{2g}.$$

- The maximum horizontal distance traveled by a projectile is called the **range**. The range  $R$  of a projectile on level ground launched at an angle  $\theta_0$  above the horizontal with initial speed  $v_0$  is given by

**Equation:**

$$R = \frac{v_0^2 \sin 2\theta_0}{g}.$$

## Conceptual Questions

**Exercise:**

**Problem:**

Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither  $0^\circ$  nor  $90^\circ$ ): (a) Is the velocity ever zero? (b) When is the velocity a minimum? A maximum? (c) Can the velocity ever be the same as the initial velocity at a time other than at  $t = 0$ ? (d) Can the speed ever be the same as the initial speed at a time other than at  $t = 0$ ?

**Exercise:**

**Problem:**

Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither  $0^\circ$  nor  $90^\circ$ ): (a) Is the acceleration ever zero? (b) Is the acceleration ever in the same direction as a component of velocity? (c) Is the acceleration ever opposite in direction to a component of velocity?

**Exercise:**

**Problem:**

For a fixed initial speed, the range of a projectile is determined by the angle at which it is fired. For all but the maximum, there are two angles that give the same range. Considering factors that might affect the ability of an archer to hit a target, such as wind, explain why the smaller angle (closer to the horizontal) is preferable. When would it be necessary for the archer to use the larger angle? Why does the punter in a football game use the higher trajectory?

**Exercise:**

**Problem:**

During a lecture demonstration, a professor places two coins on the edge of a table. She then flicks one of the coins horizontally off the table, simultaneously nudging the other over the edge. Describe the subsequent motion of the two coins, in particular discussing whether they hit the floor at the same time.

**Problems & Exercises****Exercise:****Problem:**

A projectile is launched at ground level with an initial speed of 50.0 m/s at an angle of 30.0° above the horizontal. It strikes a target above the ground 3.00 seconds later. What are the  $x$  and  $y$  distances from where the projectile was launched to where it lands?

---

**Solution:**

$$x = 1.30 \text{ m} \times 10^2$$

$$y = 30.9 \text{ m}.$$

**Exercise:****Problem:**

A ball is kicked with an initial velocity of 16 m/s in the horizontal direction and 12 m/s in the vertical direction. (a) At what speed does the ball hit the ground? (b) For how long does the ball remain in the air? (c) What maximum height is attained by the ball?

**Exercise:****Problem:**

A ball is thrown horizontally from the top of a 60.0-m building and lands 100.0 m from the base of the building. Ignore air resistance. (a) How long is the ball in the air? (b) What must have been the initial horizontal component of the velocity? (c) What is the vertical component of the velocity just before the ball hits the ground? (d) What is the velocity (including both the horizontal and vertical components) of the ball just before it hits the ground?

---

**Solution:**

(a) 3.50 s

(b) 28.6 m/s (c) 34.3 m/s

(d) 44.7 m/s, 50.2° below horizontal

**Exercise:****Problem:**

(a) A daredevil is attempting to jump his motorcycle over a line of buses parked end to end by driving up a 32° ramp at a speed of 40.0 m/s (144 km/h). How many buses can he clear if the top of the takeoff ramp is at the same height as the bus tops and the buses are 20.0 m long? (b) Discuss what your answer implies about the margin of error in this act—that is, consider how much greater the range is than the horizontal distance he must travel to miss the end of the last bus. (Neglect air resistance.)

**Exercise:**

**Problem:**

An archer shoots an arrow at a 75.0 m distant target; the bull's-eye of the target is at same height as the release height of the arrow. (a) At what angle must the arrow be released to hit the bull's-eye if its initial speed is 35.0 m/s? In this part of the problem, explicitly show how you follow the steps involved in solving projectile motion problems. (b) There is a large tree halfway between the archer and the target with an overhanging horizontal branch 3.50 m above the release height of the arrow. Will the arrow go over or under the branch?

---

**Solution:**

(a)  $18.4^\circ$

(b) The arrow will go over the branch.

**Exercise:****Problem:**

A rugby player passes the ball 7.00 m across the field, where it is caught at the same height as it left his hand. (a) At what angle was the ball thrown if its initial speed was 12.0 m/s, assuming that the smaller of the two possible angles was used? (b) What other angle gives the same range, and why would it not be used? (c) How long did this pass take?

**Exercise:**

**Problem:** Verify the ranges for the projectiles in [\[link\]](#)(a) for  $\theta = 45^\circ$  and the given initial velocities.

---

**Solution:**

$$R = \frac{v_0^2}{\sin 2\theta_0 g}$$

$$\text{For } \theta = 45^\circ, \quad R = \frac{v_0^2}{g}$$

$$R = 91.8 \text{ m for } v_0 = 30 \text{ m/s}; \quad R = 163 \text{ m for } v_0 = 40 \text{ m/s}; \quad R = 255 \text{ m for } v_0 = 50 \text{ m/s}.$$

**Exercise:****Problem:**

Verify the ranges shown for the projectiles in [\[link\]](#)(b) for an initial velocity of 50 m/s at the given initial angles.

**Exercise:****Problem:**

The cannon on a battleship can fire a shell a maximum distance of 32.0 km. (a) Calculate the initial velocity of the shell. (b) What maximum height does it reach? (At its highest, the shell is above 60% of the atmosphere—but air resistance is not really negligible as assumed to make this problem easier.) (c) The ocean is not flat, because the Earth is curved. Assume that the radius of the Earth is  $6.37 \times 10^3$  km. How many meters lower will its surface be 32.0 km from the ship along a horizontal line parallel to the surface at the ship? Does your answer imply that error introduced by the assumption of a flat Earth in projectile motion is significant here?

---

**Solution:**

(a) 560 m/s

(b)  $8.00 \times 10^3$  m

(c) 80.0 m. This error is not significant because it is only 1% of the answer in part (b).

**Exercise:**

**Problem:**

An arrow is shot from a height of 1.5 m toward a cliff of height  $H$ . It is shot with a velocity of 30 m/s at an angle of  $60^\circ$  above the horizontal. It lands on the top edge of the cliff 4.0 s later. (a) What is the height of the cliff? (b) What is the maximum height reached by the arrow along its trajectory? (c) What is the arrow's impact speed just before hitting the cliff?

**Exercise:**

**Problem:**

In the standing broad jump, one squats and then pushes off with the legs to see how far one can jump. Suppose the extension of the legs from the crouch position is 0.600 m and the acceleration achieved from this position is 1.25 times the acceleration due to gravity,  $g$ . How far can they jump? State your assumptions. (Increased range can be achieved by swinging the arms in the direction of the jump.)

---

**Solution:**

1.50 m, assuming launch angle of  $45^\circ$

**Exercise:**

**Problem:**

The world long jump record is 8.95 m (Mike Powell, USA, 1991). Treated as a projectile, what is the maximum range obtainable by a person if he has a take-off speed of 9.5 m/s? State your assumptions.

**Exercise:**

**Problem:**

Serving at a speed of 170 km/h, a tennis player hits the ball at a height of 2.5 m and an angle  $\theta$  below the horizontal. The base line is 11.9 m from the net, which is 0.91 m high. What is the angle  $\theta$  such that the ball just crosses the net? Will the ball land in the service box, whose service line is 6.40 m from the net?

---

**Solution:**

$$\theta = 6.1^\circ$$

yes, the ball lands at 5.3 m from the net

**Exercise:**

**Problem:**

A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. (a) If the ball is thrown at an angle of  $25^\circ$  relative to the ground and is caught at the same height as it is released, what is its initial speed relative to the ground? (b) How long does it take to get to the receiver? (c) What is its maximum height above its point of release?

**Exercise:**

**Problem:**

Gun sights are adjusted to aim high to compensate for the effect of gravity, effectively making the gun accurate only for a specific range. (a) If a gun is sighted to hit targets that are at the same height as the gun and 100.0 m away, how low will the bullet hit if aimed directly at a target 150.0 m away? The muzzle velocity of the bullet is 275 m/s. (b) Discuss qualitatively how a larger muzzle velocity would affect this problem and what would be the effect of air resistance.

---

**Solution:**

(a)  $-0.486\text{ m}$

(b) The larger the muzzle velocity, the smaller the deviation in the vertical direction, because the time of flight would be smaller. Air resistance would have the effect of decreasing the time of flight, therefore increasing the vertical deviation.

**Exercise:****Problem:**

An eagle is flying horizontally at a speed of  $3.00\text{ m/s}$  when the fish in her talons wiggles loose and falls into the lake  $5.00\text{ m}$  below. Calculate the velocity of the fish relative to the water when it hits the water.

**Exercise:****Problem:**

An owl is carrying a mouse to the chicks in its nest. Its position at that time is  $4.00\text{ m}$  west and  $12.0\text{ m}$  above the center of the  $30.0\text{ cm}$  diameter nest. The owl is flying east at  $3.50\text{ m/s}$  at an angle  $30.0^\circ$  below the horizontal when it accidentally drops the mouse. Is the owl lucky enough to have the mouse hit the nest? To answer this question, calculate the horizontal position of the mouse when it has fallen  $12.0\text{ m}$ .

---

**Solution:**

$4.23\text{ m}$ . No, the owl is not lucky; he misses the nest.

**Exercise:****Problem:**

Suppose a soccer player kicks the ball from a distance  $30\text{ m}$  toward the goal. Find the initial speed of the ball if it just passes over the goal,  $2.4\text{ m}$  above the ground, given the initial direction to be  $40^\circ$  above the horizontal.

**Exercise:****Problem:**

Can a goalkeeper at her/ his goal kick a soccer ball into the opponent's goal without the ball touching the ground? The distance will be about  $95\text{ m}$ . A goalkeeper can give the ball a speed of  $30\text{ m/s}$ .

---

**Solution:**

No, the maximum range (neglecting air resistance) is about  $92\text{ m}$ .

**Exercise:****Problem:**

The free throw line in basketball is  $4.57\text{ m}$  ( $15\text{ ft}$ ) from the basket, which is  $3.05\text{ m}$  ( $10\text{ ft}$ ) above the floor. A player standing on the free throw line throws the ball with an initial speed of  $8.15\text{ m/s}$ , releasing it at a height of  $2.44\text{ m}$  ( $8\text{ ft}$ ) above the floor. At what angle above the horizontal must the ball be thrown to exactly hit the basket? Note that most players will use a large initial angle rather than a flat shot because it allows for a larger margin of error. Explicitly show how you follow the steps involved in solving projectile motion problems.

**Exercise:**



**Problem:**

In 2007, Michael Carter (U.S.) set a world record in the shot put with a throw of 24.77 m. What was the initial speed of the shot if he released it at a height of 2.10 m and threw it at an angle of  $38.0^\circ$  above the horizontal? (Although the maximum distance for a projectile on level ground is achieved at  $45^\circ$  when air resistance is neglected, the actual angle to achieve maximum range is smaller; thus,  $38^\circ$  will give a longer range than  $45^\circ$  in the shot put.)

**Solution:**

15.0 m/s

**Exercise:****Problem:**

A basketball player is running at 5.00 m/s directly toward the basket when he jumps into the air to dunk the ball. He maintains his horizontal velocity. (a) What vertical velocity does he need to rise 0.750 m above the floor? (b) How far from the basket (measured in the horizontal direction) must he start his jump to reach his maximum height at the same time as he reaches the basket?

**Exercise:****Problem:**

A football player punts the ball at a  $45.0^\circ$  angle. Without an effect from the wind, the ball would travel 60.0 m horizontally. (a) What is the initial speed of the ball? (b) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.50 m/s. What distance does the ball travel horizontally?

**Solution:**

(a) 24.2 m/s

(b) The ball travels a total of 57.4 m with the brief gust of wind.

**Exercise:****Problem:**

Prove that the trajectory of a projectile is parabolic, having the form  $y = ax + bx^2$ . To obtain this expression, solve the equation  $x = v_{0x}t$  for  $t$  and substitute it into the expression for  $y = v_{0y}t - (1/2)gt^2$  (These equations describe the  $x$  and  $y$  positions of a projectile that starts at the origin.) You should obtain an equation of the form  $y = ax + bx^2$  where  $a$  and  $b$  are constants.

**Exercise:****Problem:**

Derive  $R = \frac{v_0^2 \sin 2\theta_0}{g}$  for the range of a projectile on level ground by finding the time  $t$  at which  $y$  becomes zero and substituting this value of  $t$  into the expression for  $x - x_0$ , noting that  $R = x - x_0$

**Solution:**

$$y - y_0 = 0 = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin \theta)t - \frac{1}{2}gt^2,$$

$$\text{so that } t = \frac{2(v_0 \sin \theta)}{g}$$

$x - x_0 = v_{0x}t = (v_0 \cos \theta)t = R$ , and substituting for  $t$  gives:

$$R = v_0 \cos \theta \left( \frac{2v_0 \sin \theta}{g} \right) = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

since  $2 \sin \theta \cos \theta = \sin 2\theta$ , the range is:

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

### Exercise:

#### Problem:

**Unreasonable Results** (a) Find the maximum range of a super cannon that has a muzzle velocity of 4.0 km/s. (b) What is unreasonable about the range you found? (c) Is the premise unreasonable or is the available equation inapplicable? Explain your answer. (d) If such a muzzle velocity could be obtained, discuss the effects of air resistance, thinning air with altitude, and the curvature of the Earth on the range of the super cannon.

### Exercise:

#### Problem:

**Construct Your Own Problem** Consider a ball tossed over a fence. Construct a problem in which you calculate the ball's needed initial velocity to just clear the fence. Among the things to determine are; the height of the fence, the distance to the fence from the point of release of the ball, and the height at which the ball is released. You should also consider whether it is possible to choose the initial speed for the ball and just calculate the angle at which it is thrown. Also examine the possibility of multiple solutions given the distances and heights you have chosen.

## Glossary

air resistance

a frictional force that slows the motion of objects as they travel through the air; when solving basic physics problems, air resistance is assumed to be zero

kinematics

the study of motion without regard to mass or force

motion

displacement of an object as a function of time

projectile

an object that travels through the air and experiences only acceleration due to gravity

projectile motion

the motion of an object that is subject only to the acceleration of gravity

range

the maximum horizontal distance that a projectile travels

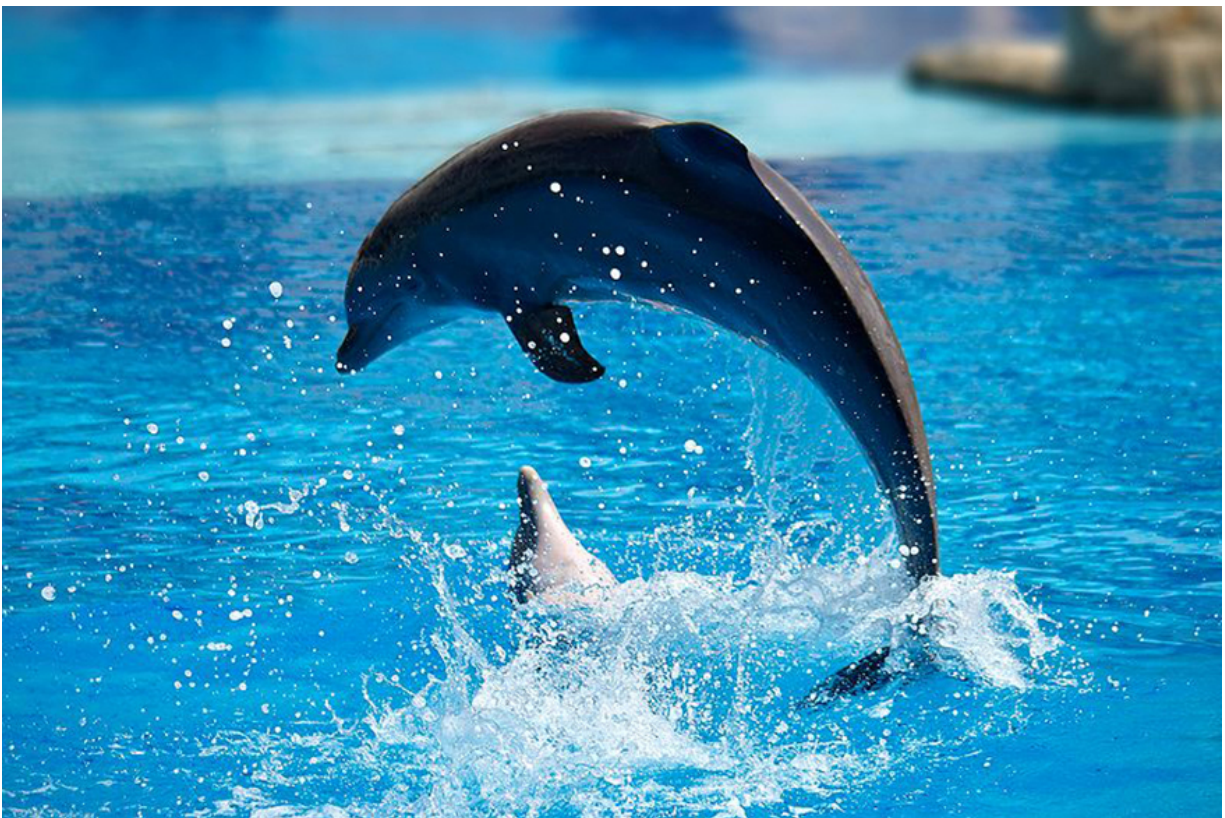
trajectory

the path of a projectile through the air

## Introduction to Dynamics: Newton's Laws of Motion

class="introduction"

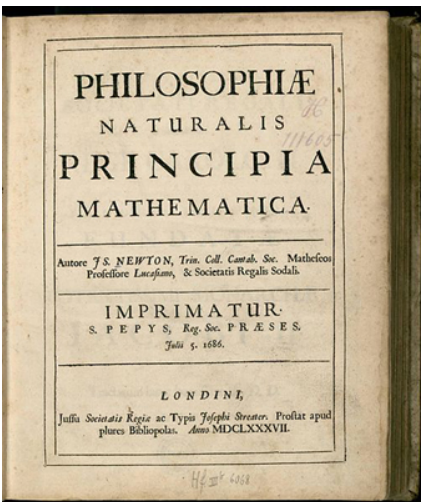
Newton's laws of motion describe the motion of the dolphin's path.  
(credit: Jin Jang)



Motion draws our attention. Motion itself can be beautiful, causing us to marvel at the forces needed to achieve spectacular motion, such as that of a

dolphin jumping out of the water, or a pole vaulter, or the flight of a bird, or the orbit of a satellite. The study of motion is kinematics, but kinematics only *describes* the way objects move—their velocity and their acceleration. **Dynamics** considers the forces that affect the motion of moving objects and systems. Newton's laws of motion are the foundation of dynamics. These laws provide an example of the breadth and simplicity of principles under which nature functions. They are also universal laws in that they apply to similar situations on Earth as well as in space.

Isaac Newton's (1642–1727) laws of motion were just one part of the monumental work that has made him legendary. The development of Newton's laws marks the transition from the Renaissance into the modern era. This transition was characterized by a revolutionary change in the way people thought about the physical universe. For many centuries natural philosophers had debated the nature of the universe based largely on certain rules of logic with great weight given to the thoughts of earlier classical philosophers such as Aristotle (384–322 BC). Among the many great thinkers who contributed to this change were Newton and Galileo.



Isaac Newton's  
monumental work,  
*Philosophiæ  
Naturalis Principia  
Mathematica*, was  
published in 1687. It  
proposed scientific

laws that are still  
used today to  
describe the motion  
of objects. (credit:  
Service commun de  
la documentation de  
l'Université de  
Strasbourg)

Galileo was instrumental in establishing *observation* as the absolute determinant of truth, rather than “logical” argument. Galileo’s use of the telescope was his most notable achievement in demonstrating the importance of observation. He discovered moons orbiting Jupiter and made other observations that were inconsistent with certain ancient ideas and religious dogma. For this reason, and because of the manner in which he dealt with those in authority, Galileo was tried by the Inquisition and punished. He spent the final years of his life under a form of house arrest. Because others before Galileo had also made discoveries by *observing* the nature of the universe, and because repeated observations verified those of Galileo, his work could not be suppressed or denied. After his death, his work was verified by others, and his ideas were eventually accepted by the church and scientific communities.

Galileo also contributed to the formation of what is now called Newton’s first law of motion. Newton made use of the work of his predecessors, which enabled him to develop laws of motion, discover the law of gravity, invent calculus, and make great contributions to the theories of light and color. It is amazing that many of these developments were made with Newton working alone, without the benefit of the usual interactions that take place among scientists today.

It was not until the advent of modern physics early in the 20th century that it was discovered that Newton’s laws of motion produce a good approximation to motion only when the objects are moving at speeds much, much less than the speed of light and when those objects are larger than the

size of most molecules (about  $10^{-9}$  m in diameter). These constraints define the realm of classical mechanics, as discussed in [Introduction to the Nature of Science and Physics](#). At the beginning of the 20<sup>th</sup> century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, developed quantum theory. This theory does not have the constraints present in classical physics. All of the situations we consider in this chapter, and all those preceding the introduction of relativity in [Special Relativity](#), are in the realm of classical physics.

**Note:**

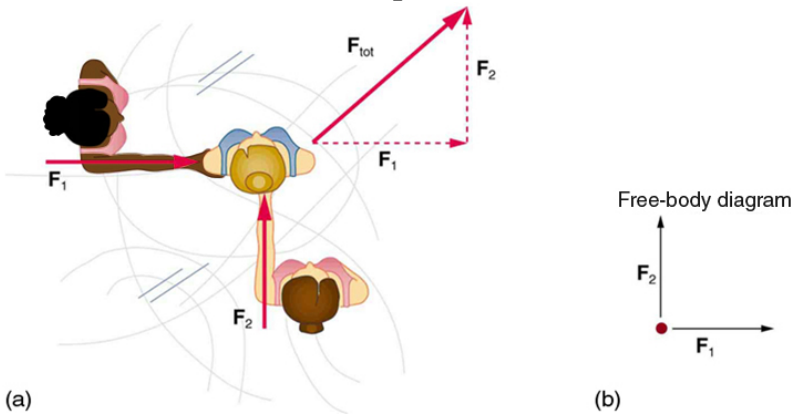
**Making Connections: Past and Present Philosophy**

*The importance of observation* and the concept of *cause and effect* were not always so entrenched in human thinking. This realization was a part of the evolution of modern physics from natural philosophy. The achievements of Galileo, Newton, Einstein, and others were key milestones in the history of scientific thought. Most of the scientific theories that are described in this book descended from the work of these scientists.

## Development of Force Concept

- Understand the definition of force.

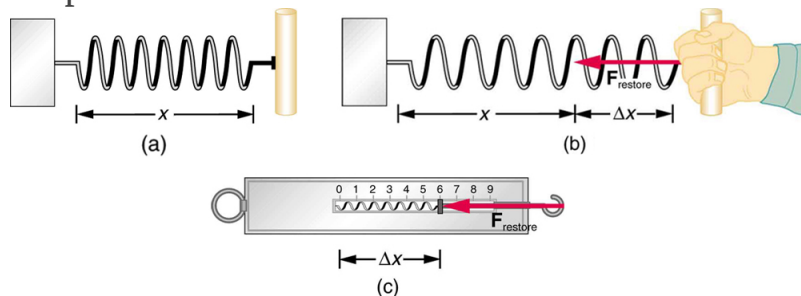
**Dynamics** is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of **force**—that is, a push or a pull—is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard. For example, a cannon exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in [\[link\]](#), we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors, as illustrated in [\[link\]](#)(a) for two ice skaters. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or by trigonometric methods. These ideas were developed in [Two-Dimensional Kinematics](#).



Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like other vectors, so the total force on the third skater is in the direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.

[\[link\]](#)(b) is our first example of a **free-body diagram**, which is a technique used to illustrate all the **external forces** acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting *on* the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton's laws of motion.

A more quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard distance. One possibility is to stretch a spring a certain fixed distance, as illustrated in [\[link\]](#), and use the force it exerts to pull itself back to its relaxed shape—called a *restoring force*—as a standard. The magnitude of all other forces can be stated as multiples of this standard unit of force. Many other possibilities exist for standard forces. (One that we will encounter in [Magnetism](#) is the magnetic force between two wires carrying electric current.) Some alternative definitions of force will be given later in this chapter.



The force exerted by a stretched spring can be used as a standard unit of force. (a) This spring has a length  $x$  when undistorted. (b) When stretched a distance  $\Delta x$ , the spring exerts a restoring force,  $\mathbf{F}_{\text{restore}}$ , which is reproducible. (c) A spring scale is one device that uses a spring to measure force. The force  $\mathbf{F}_{\text{restore}}$  is exerted on whatever is attached to the hook. Here  $\mathbf{F}_{\text{restore}}$  has a



magnitude of 6 units in the force standard being employed.

**Note:**

**Take-Home Experiment: Force Standards**

To investigate force standards and cause and effect, get two identical rubber bands. Hang one rubber band vertically on a hook. Find a small household item that could be attached to the rubber band using a paper clip, and use this item as a weight to investigate the stretch of the rubber band. Measure the amount of stretch produced in the rubber band with one, two, and four of these (identical) items suspended from the rubber band. What is the relationship between the number of items and the amount of stretch? How large a stretch would you expect for the same number of items suspended from two rubber bands? What happens to the amount of stretch of the rubber band (with the weights attached) if the weights are also pushed to the side with a pencil?

## Section Summary

- **Dynamics** is the study of how forces affect the motion of objects.
- **Force** is a push or pull that can be defined in terms of various standards, and it is a vector having both magnitude and direction.
- **External forces** are any outside forces that act on a body. A **free-body diagram** is a drawing of all external forces acting on a body.

## Conceptual Questions

**Exercise:**

**Problem:**

Propose a force standard different from the example of a stretched spring discussed in the text. Your standard must be capable of producing the same force repeatedly.

**Exercise:****Problem:**

What properties do forces have that allow us to classify them as vectors?

**Glossary**

dynamics

the study of how forces affect the motion of objects and systems

external force

a force acting on an object or system that originates outside of the object or system

free-body diagram

a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot

force

a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force

## Newton's First Law of Motion: Inertia

- Define mass and inertia.
- Understand Newton's first law of motion.

Experience suggests that an object at rest will remain at rest if left alone, and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. What **Newton's first law of motion** states, however, is the following:

### **Note:**

#### **Newton's First Law of Motion**

A body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

Note the repeated use of the verb “remains.” We can think of this law as preserving the status quo of motion.

Rather than contradicting our experience, **Newton's first law of motion** states that there must be a *cause* (which is a net external force) *for there to be any change in velocity (either a change in magnitude or direction)*. We will define *net external force* in the next section. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappeared, would the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the *cause* of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were

completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Newton's first law is completely general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have thoroughly verified that any change in velocity (speed or direction) must be caused by an external force. The idea of *generally applicable or universal laws* is important not only here—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law, and Newton, who clarified it, was to ask the fundamental question, “What is the cause?” Thinking in terms of cause and effect is a worldview fundamentally different from the typical ancient Greek approach when questions such as “Why does a tiger have stripes?” would have been answered in Aristotelian fashion, “That is the nature of the beast.” True perhaps, but not a useful insight.

## Mass

The property of a body to remain at rest or to remain in motion with constant velocity is called **inertia**. Newton's first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its **mass**. Roughly speaking, mass is a measure of the amount of “stuff” (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of various types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is very difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this

manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

**Exercise:**

**Check Your Understanding**

**Problem:**

Which has more mass: a kilogram of cotton balls or a kilogram of gold?

---

**Solution:**

**Answer**

They are equal. A kilogram of one substance is equal in mass to a kilogram of another substance. The quantities that might differ between them are volume and density.

**Section Summary**

- **Newton's first law of motion** states that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force. This is also known as the **law of inertia**.
- **Inertia** is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object's mass.
- **Mass** is the quantity of matter in a substance.

**Conceptual Questions**

**Exercise:**

**Problem:** How are inertia and mass related?

**Exercise:**

**Problem:**

What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?

## **Glossary**

inertia

the tendency of an object to remain at rest or remain in motion

law of inertia

see Newton's first law of motion

mass

the quantity of matter in a substance; measured in kilograms

Newton's first law of motion

a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia

## The First Condition for Equilibrium

- State the first condition of equilibrium.
- Explain static equilibrium.
- Explain dynamic equilibrium.

The first condition necessary to achieve equilibrium is the one already mentioned: the net external force on the system must be zero. Expressed as an equation, this is simply

**Equation:**

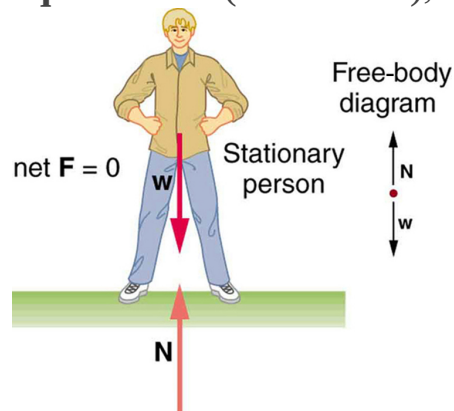
$$\text{net } \mathbf{F} = 0$$

Note that if net  $F$  is zero, then the net external force in *any* direction is zero. For example, the net external forces along the typical x- and y-axes are zero. This is written as

**Equation:**

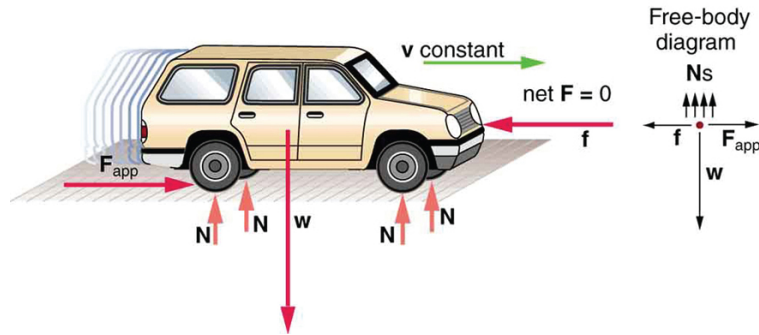
$$\text{net } F_x = 0 \text{ and } F_y = 0$$

[\[link\]](#) and [\[link\]](#) illustrate situations where net  $F = 0$  for both **static equilibrium** (motionless), and **dynamic equilibrium** (constant velocity).



This motionless person is in static equilibrium. The forces acting on him add up to zero. Both

forces are vertical in this case.

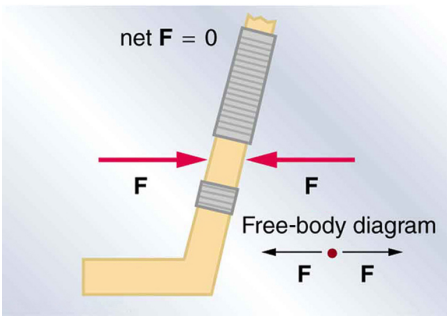


This car is in dynamic equilibrium because it is moving at constant velocity. There are horizontal and vertical forces, but the net external force in any direction is zero. The applied force  $F_{\text{app}}$  between the tires and the road is balanced by air friction, and the weight of the car is supported by the normal forces, here shown to be equal for all four tires.

However, it is not sufficient for the net external force of a system to be zero for a system to be in equilibrium. Consider the two situations illustrated in [\[link\]](#) and [\[link\]](#) where forces are applied to an ice hockey stick lying flat on ice. The net external force is zero in both situations shown in the figure; but in one case, equilibrium is achieved, whereas in the other, it is not. In [\[link\]](#), the ice hockey stick remains motionless. But in [\[link\]](#), with the same forces applied in different places, the stick experiences accelerated rotation. Therefore, we know that the point at which a force is applied is another factor in determining whether or not equilibrium is achieved. This will be explored further in the next section.



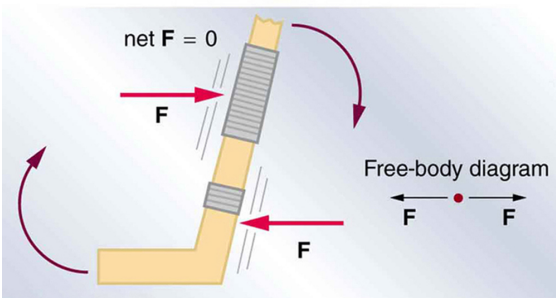
Equilibrium: remains stationary



An ice hockey stick lying flat on ice with two equal and opposite horizontal forces applied to it. Friction is negligible, and the gravitational force is balanced by the support of the ice (a normal force). Thus, net  $F = 0$ .

Equilibrium is achieved, which is static equilibrium in this case.

Nonequilibrium: rotation accelerates



The same forces are applied at other points and the stick

rotates—in fact, it experiences an accelerated rotation. Here net  $F = 0$  but the system is *not* at equilibrium. Hence, the net  $F = 0$  is a necessary—but not sufficient—condition for achieving equilibrium.

**Note:**

PhET Explorations: Torque

Investigate how torque causes an object to rotate. Discover the relationships between angular acceleration, moment of inertia, angular momentum and torque.

[Torqu  
e](#)

## Section Summary

- Statics is the study of forces in equilibrium.
- Two conditions must be met to achieve equilibrium, which is defined to be motion without linear or rotational acceleration.
- The first condition necessary to achieve equilibrium is that the net external force on the system must be zero, so that net  $\mathbf{F} = 0$ .

## Conceptual Questions

**Exercise:****Problem:**

What can you say about the velocity of a moving body that is in dynamic equilibrium? Draw a sketch of such a body using clearly labeled arrows to represent all external forces on the body.

**Exercise:****Problem:**

Under what conditions can a rotating body be in equilibrium? Give an example.

**Glossary**

static equilibrium

a state of equilibrium in which the net external force and torque acting on a system is zero

dynamic equilibrium

a state of equilibrium in which the net external force and torque on a system moving with constant velocity are zero

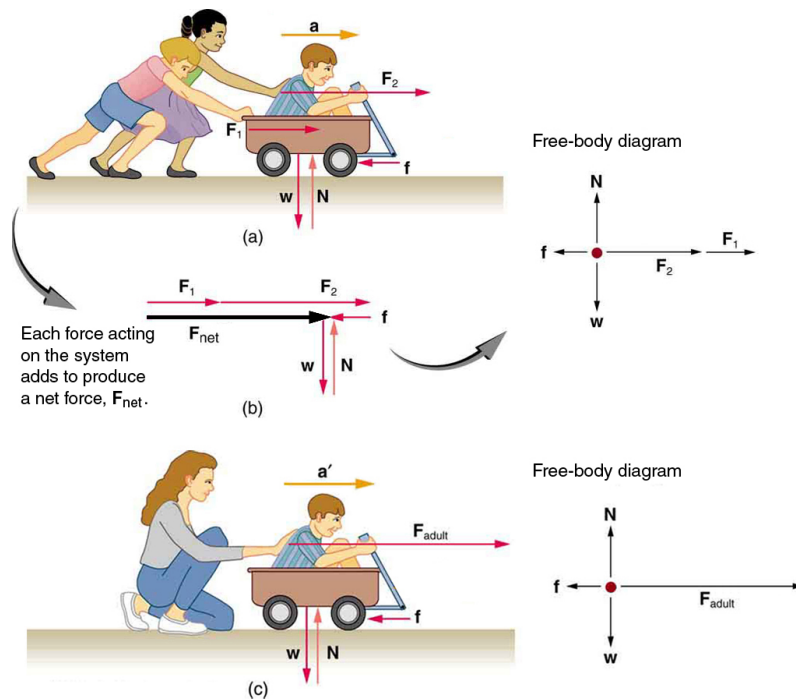
## Newton's Second Law of Motion: Concept of a System

- Define net force, external force, and system.
- Understand Newton's second law of motion.
- Apply Newton's second law to determine the weight of an object.

**Newton's second law of motion** is closely related to Newton's first law of motion. It mathematically states the cause and effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an **acceleration**. Newton's first law says that a net external force causes a change in motion; thus, we see that a *net external force causes acceleration*.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an **external force** acts from outside the **system** of interest. For example, in [\[link\]](#)(a) the system of interest is the wagon plus the child in it. The two forces exerted by the other children are external forces. An internal force acts between elements of the system. Again looking at [\[link\]](#)(a), the force the child in the wagon exerts to hang onto the wagon is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton's first law. (The internal forces actually cancel, as we shall see in the next section.) *You must define the boundaries of the system before you can determine which forces are external.* Sometimes the system is obvious, whereas other times identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton's laws. This concept will be revisited many times on our journey through physics.



Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight  $w$  of the system and the support of the ground  $N$  are also shown for completeness and are assumed to cancel. The vector  $f$  represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system add together to produce a net force,  $F_{\text{net}}$ . The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger

acceleration ( $\mathbf{a}' > \mathbf{a}$ ) when an adult pushes the child.

Now, it seems reasonable that acceleration should be directly proportional to and in the same direction as the net (total) external force acting on a system. This assumption has been verified experimentally and is illustrated in [\[link\]](#). In part (a), a smaller force causes a smaller acceleration than the larger force illustrated in part (c). For completeness, the vertical forces are also shown; they are assumed to cancel since there is no acceleration in the vertical direction. The vertical forces are the weight  $\mathbf{w}$  and the support of the ground  $\mathbf{N}$ , and the horizontal force  $\mathbf{f}$  represents the force of friction. These will be discussed in more detail in later sections. For now, we will define **friction** as a force that opposes the motion past each other of objects that are touching. [\[link\]](#)(b) shows how vectors representing the external forces add together to produce a net force,  $\mathbf{F}_{\text{net}}$ .

To obtain an equation for Newton's second law, we first write the relationship of acceleration and net external force as the proportionality **Equation:**

$$\mathbf{a} \propto \mathbf{F}_{\text{net}},$$

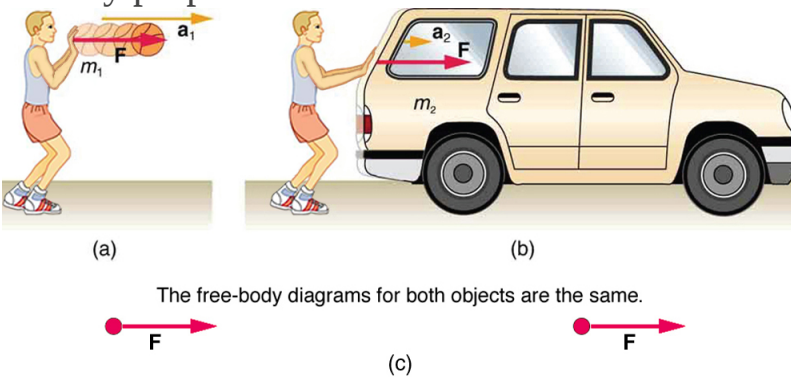
where the symbol  $\propto$  means “proportional to,” and  $\mathbf{F}_{\text{net}}$  is the **net external force**. (The net external force is the vector sum of all external forces and can be determined graphically, using the head-to-tail method, or analytically, using components. The techniques are the same as for the addition of other vectors, and are covered in [Two-Dimensional Kinematics](#).) This proportionality states what we have said in words—*acceleration is directly proportional to the net external force*. Once the system of interest is chosen, it is important to identify the external forces and ignore the internal ones. It is a tremendous simplification not to have to consider the numerous internal forces acting between objects within the system, such as muscular forces within the child's body, let alone the myriad of forces between atoms in the objects, but by doing so, we can easily solve some very complex problems with only minimal error due to our simplification

Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. And indeed, as illustrated in [\[link\]](#), the same net external force applied to a car produces a much smaller acceleration than when applied to a basketball. The proportionality is written as

**Equation:**

$$\mathbf{a} \propto \frac{1}{m}$$

where  $m$  is the mass of the system. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is exactly linearly proportional to the net external force.



The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (The effect of gravity on the ball is ignored.) (b) The same player exerts an identical force on a stalled SUV and produces a far smaller acceleration (even if friction is negligible). (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for the free-body diagram will emerge as you do more problems.

It has been found that the acceleration of an object depends *only* on the net external force and the mass of the object. Combining the two proportionalities just given yields Newton's second law of motion.

**Note:**

**Newton's Second Law of Motion**

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

In equation form, Newton's second law of motion is

**Equation:**

$$\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}.$$

This is often written in the more familiar form

**Equation:**

$$\mathbf{F}_{\text{net}} = m\mathbf{a}.$$

When only the magnitude of force and acceleration are considered, this equation is simply

**Equation:**

$$F_{\text{net}} = ma.$$

Although these last two equations are really the same, the first gives more insight into what Newton's second law means. The law is a *cause and effect relationship* among three quantities that is not simply based on their definitions. The validity of the second law is completely based on experimental verification.



## Units of Force

$\mathbf{F}_{\text{net}} = m\mathbf{a}$  is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the **newton** (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of  $1\text{m/s}^2$ . That is, since  $\mathbf{F}_{\text{net}} = m\mathbf{a}$ ,

**Equation:**

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2.$$

While almost the entire world uses the newton for the unit of force, in the United States the most familiar unit of force is the pound (lb), where  $1 \text{ N} = 0.225 \text{ lb}$ .

## Weight and the Gravitational Force

When an object is dropped, it accelerates toward the center of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight  $\mathbf{w}$** . Weight can be denoted as a vector  $\mathbf{w}$  because it has a direction; *down* is, by definition, the direction of gravity, and hence weight is a downward force. The magnitude of weight is denoted as  $w$ . Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration  $g$ . Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass  $m$  falling downward toward Earth. It experiences only the downward force of gravity, which has magnitude  $w$ . Newton's second law states that the magnitude of the net external force on an object is  $F_{\text{net}} = ma$ .

Since the object experiences only the downward force of gravity,  $F_{\text{net}} = w$ . We know that the acceleration of an object due to gravity is  $g$ , or  $a = g$ . Substituting these into Newton's second law gives

**Note:****Weight**

This is the equation for *weight*—the gravitational force on a mass  $m$ :

**Equation:**

$$w = mg.$$

Since  $g = 9.80 \text{ m/s}^2$  on Earth, the weight of a 1.0 kg object on Earth is 9.8 N, as we see:

**Equation:**

$$w = mg = (1.0 \text{ kg})(9.80 \text{ m/s}^2) = 9.8 \text{ N}.$$

Recall that  $g$  can take a positive or negative value, depending on the positive direction in the coordinate system. Be sure to take this into consideration when solving problems with weight.

When the net external force on an object is its weight, we say that it is in **free-fall**. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

The acceleration due to gravity  $g$  varies slightly over the surface of Earth, so that the weight of an object depends on location and is not an intrinsic property of the object. Weight varies dramatically if one leaves Earth's surface. On the Moon, for example, the acceleration due to gravity is only  $1.67 \text{ m/s}^2$ . A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that *the weight of an object is the gravitational force on it from the nearest large body*, such as Earth, the Moon, the Sun, and so on. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of “weightlessness” and

“microgravity,” they are really referring to the phenomenon we call “free-fall” in physics. We shall use the above definition of weight, and we will make careful distinctions between free-fall and actual weightlessness.

It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much “stuff”) and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object. Furthermore, the terms *mass* and *weight* are used interchangeably in everyday language; for example, our medical records often show our “weight” in kilograms, but never in the correct units of newtons.

**Note:****Common Misconceptions: Mass vs. Weight**

Mass and weight are often used interchangeably in everyday language. However, in science, these terms are distinctly different from one another. Mass is a measure of how much matter is in an object. The typical measure of mass is the kilogram (or the “slug” in English units). Weight, on the other hand, is a measure of the force of gravity acting on an object. Weight is equal to the mass of an object ( $m$ ) multiplied by the acceleration due to gravity ( $g$ ). Like any other force, weight is measured in terms of newtons (or pounds in English units).

Assuming the mass of an object is kept intact, it will remain the same, regardless of its location. However, because weight depends on the acceleration due to gravity, the weight of an object *can change* when the object enters into a region with stronger or weaker gravity. For example, the acceleration due to gravity on the Moon is  $1.67 \text{ m/s}^2$  (which is much less than the acceleration due to gravity on Earth,  $9.80 \text{ m/s}^2$ ). If you measured your weight on Earth and then measured your weight on the Moon, you would find that you “weigh” much less, even though you do not look any skinnier. This is because the force of gravity is weaker on the Moon. In fact, when people say that they are “losing weight,” they really

mean that they are losing “mass” (which in turn causes them to weigh less).

**Note:**

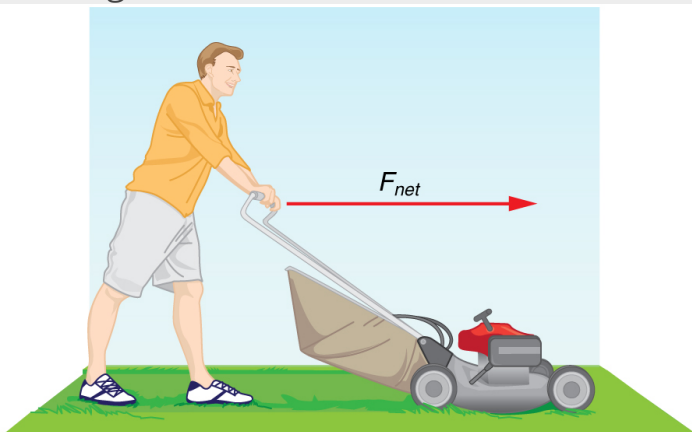
**Take-Home Experiment: Mass and Weight**

What do bathroom scales measure? When you stand on a bathroom scale, what happens to the scale? It depresses slightly. The scale contains springs that compress in proportion to your weight—similar to rubber bands expanding when pulled. The springs provide a measure of your weight (for an object which is not accelerating). This is a force in newtons (or pounds). In most countries, the measurement is divided by 9.80 to give a reading in mass units of kilograms. The scale measures weight but is calibrated to provide information about mass. While standing on a bathroom scale, push down on a table next to you. What happens to the reading? Why? Would your scale measure the same “mass” on Earth as on the Moon?

**Example:**

**What Acceleration Can a Person Produce when Pushing a Lawn Mower?**

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration?



The net force on a lawn mower is 51

N to the right. At what rate does the lawn mower accelerate to the right?

**Strategy**

Since  $\mathbf{F}_{\text{net}}$  and  $m$  are given, the acceleration can be calculated directly from Newton's second law as stated in  $\mathbf{F}_{\text{net}} = m\mathbf{a}$ .

**Solution**

The magnitude of the acceleration  $a$  is  $a = \frac{F_{\text{net}}}{m}$ . Entering known values gives

**Equation:**

$$a = \frac{51 \text{ N}}{24 \text{ kg}}$$

Substituting the units  $\text{kg} \cdot \text{m}/\text{s}^2$  for N yields

**Equation:**

$$a = \frac{51 \text{ kg} \cdot \text{m}/\text{s}^2}{24 \text{ kg}} = 2.1 \text{ m}/\text{s}^2.$$

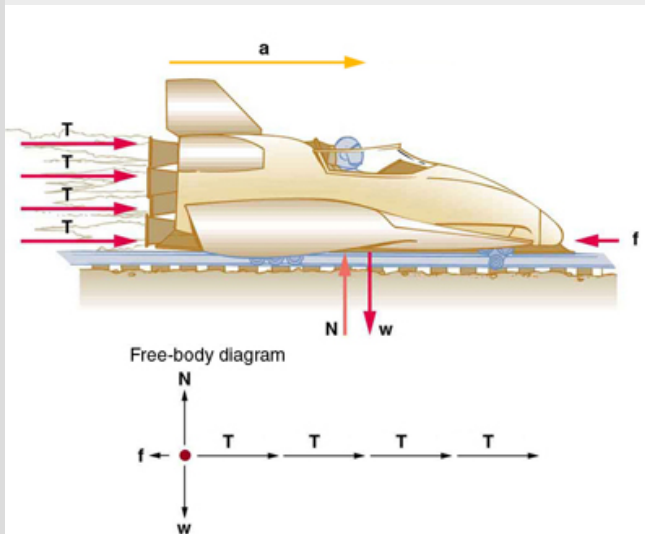
**Discussion**

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

**Example:**

### What Rocket Thrust Accelerates This Sled?

Prior to manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust  $\mathbf{T}$ , for the four-rocket propulsion system shown in [\[link\]](#). The sled's initial acceleration is  $49 \text{ m/s}^2$ , the mass of the system is  $2100 \text{ kg}$ , and the force of friction opposing the motion is known to be  $650 \text{ N}$ .



A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust  $\mathbf{T}$ . As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force  $\mathbf{N}$  on the system that is equal in magnitude and opposite in direction to its weight,  $\mathbf{w}$ . The system here is the sled, its rockets, and rider, so none of the forces *between* these objects are considered. The arrow representing friction ( $\mathbf{f}$ ) is drawn larger than scale.

**Strategy**

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem.

Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

**Solution**

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines.

Since we have defined the direction of the force and acceleration as acting "to the right," we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

**Equation:**

$$F_{\text{net}} = ma,$$

where  $F_{\text{net}}$  is the net force along the horizontal direction. We can see from [\[link\]](#) that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

**Equation:**

$$F_{\text{net}} = 4T - f.$$

Substituting this into Newton's second law gives

**Equation:**

$$F_{\text{net}} = ma = 4T - f.$$

Using a little algebra, we solve for the total thrust  $4T$ :

**Equation:**

$$4T = ma + f.$$

Substituting known values yields

**Equation:**

$$4T = ma + f = (2100 \text{ kg})(49 \text{ m/s}^2) + 650 \text{ N}.$$

So the total thrust is

**Equation:**

$$4T = 1.0 \times 10^5 \text{ N},$$

and the individual thrusts are

**Equation:**

$$T = \frac{1.0 \times 10^5 \text{ N}}{4} = 2.6 \times 10^4 \text{ N}.$$

### Discussion

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45  $g$ 's. (Recall that  $g$ , the acceleration due to gravity, is  $9.80 \text{ m/s}^2$ . When we say that an acceleration is 45  $g$ 's, it is  $45 \times 9.80 \text{ m/s}^2$ , which is approximately  $440 \text{ m/s}^2$ .) While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

### Section Summary

- Acceleration,  $\mathbf{a}$ , is defined as a change in velocity, meaning a change in its magnitude or direction, or both.
- An external force is one acting on a system from outside the system, as opposed to internal forces, which act between components within the



system.

- Newton's second law of motion states that the acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.
- In equation form, Newton's second law of motion is  $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$ .
- This is often written in the more familiar form:  $\mathbf{F}_{\text{net}} = m\mathbf{a}$ .
- The weight  $\mathbf{w}$  of an object is defined as the force of gravity acting on an object of mass  $m$ . The object experiences an acceleration due to gravity  $\mathbf{g}$ :

**Equation:**

$$\mathbf{w} = m\mathbf{g}.$$

- If the only force acting on an object is due to gravity, the object is in free fall.
- Friction is a force that opposes the motion past each other of objects that are touching.

## Conceptual Questions

**Exercise:**

**Problem:**

Which statement is correct? (a) Net force causes motion. (b) Net force causes change in motion. Explain your answer and give an example.

**Exercise:**

**Problem:**

Why can we neglect forces such as those holding a body together when we apply Newton's second law of motion?

**Exercise:**

**Problem:**

Explain how the choice of the “system of interest” affects which forces must be considered when applying Newton’s second law of motion.

**Exercise:**

**Problem:**

Describe a situation in which the net external force on a system is not zero, yet its speed remains constant.

**Exercise:**

**Problem:**

A system can have a nonzero velocity while the net external force on it is zero. Describe such a situation.

**Exercise:**

**Problem:**

A rock is thrown straight up. What is the net external force acting on the rock when it is at the top of its trajectory?

**Exercise:**

**Problem:**

(a) Give an example of different net external forces acting on the same system to produce different accelerations. (b) Give an example of the same net external force acting on systems of different masses, producing different accelerations. (c) What law accurately describes both effects? State it in words and as an equation.

**Exercise:**

**Problem:**

If the acceleration of a system is zero, are no external forces acting on it? What about internal forces? Explain your answers.

**Exercise:**

**Problem:**

If a constant, nonzero force is applied to an object, what can you say about the velocity and acceleration of the object?

**Exercise:****Problem:**

The gravitational force on the basketball in [\[link\]](#) is ignored. When gravity *is* taken into account, what is the direction of the net external force on the basketball—above horizontal, below horizontal, or still horizontal?

**Problem Exercises**

**You may assume data taken from illustrations is accurate to three digits.**

**Exercise:****Problem:**

A 63.0-kg sprinter starts a race with an acceleration of  $4.20 \text{ m/s}^2$ . What is the net external force on him?

---

**Solution:**

265 N

**Exercise:****Problem:**

If the sprinter from the previous problem accelerates at that rate for 20 m, and then maintains that velocity for the remainder of the 100-m dash, what will be his time for the race?

**Exercise:**

**Problem:**

A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of its acceleration.

---

**Solution:**

$$13.3 \text{ m/s}^2$$

**Exercise:****Problem:**

Since astronauts in orbit are apparently weightless, a clever method of measuring their masses is needed to monitor their mass gains or losses to adjust diets. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted and the astronaut's acceleration is measured to be  $0.893 \text{ m/s}^2$ . (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which they orbit experiences an equal and opposite force. Discuss how this would affect the measurement of the astronaut's acceleration. Propose a method in which recoil of the vehicle is avoided.

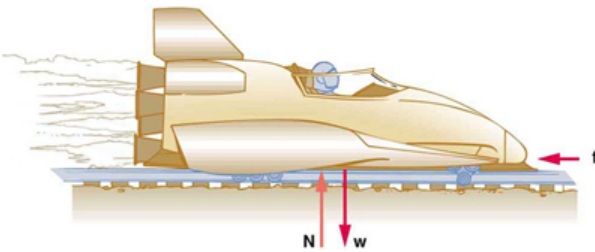
**Exercise:****Problem:**

In [\[link\]](#), the net external force on the 24-kg mower is stated to be 51 N. If the force of friction opposing the motion is 24 N, what force  $F$  (in newtons) is the person exerting on the mower? Suppose the mower is moving at 1.5 m/s when the force  $F$  is removed. How far will the mower go before stopping?

**Exercise:**

**Problem:**

The same rocket sled drawn in [\[link\]](#) is decelerated at a rate of  $196 \text{ m/s}^2$ . What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2100 kg.

**Exercise:****Problem:**

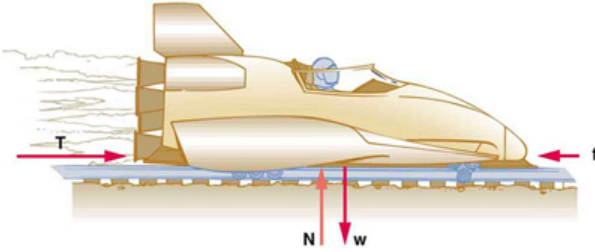
(a) If the rocket sled shown in [\[link\]](#) starts with only one rocket burning, what is the magnitude of its acceleration? Assume that the mass of the system is 2100 kg, the thrust  $T$  is  $2.4 \times 10^4 \text{ N}$ , and the force of friction opposing the motion is known to be 650 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?

---

**Solution:**

(a)  $12 \text{ m/s}^2$ .

(b) The acceleration is not one-fourth of what it was with all rockets burning because the frictional force is still as large as it was with all rockets burning.



### Exercise:

#### Problem:

What is the deceleration of the rocket sled if it comes to rest in 1.1 s from a speed of 1000 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)

### Exercise:

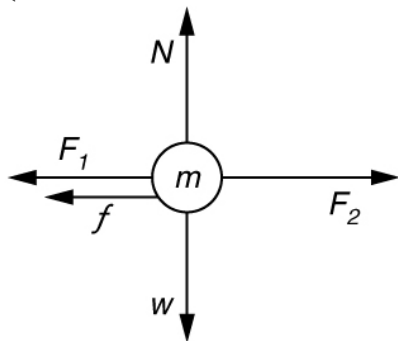
#### Problem:

Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg. (a) What is the system of interest if the acceleration of the child in the wagon is to be calculated? (b) Draw a free-body diagram, including all forces acting on the system. (c) Calculate the acceleration. (d) What would the acceleration be if friction were 15.0 N?

#### Solution:

(a) The system is the child in the wagon plus the wagon.

(b)



(c)  $a = 0.130 \text{ m/s}^2$  in the direction of the second child's push.

(d)  $a = 0.00 \text{ m/s}^2$

**Exercise:**

**Problem:**

A powerful motorcycle can produce an acceleration of  $3.50 \text{ m/s}^2$  while traveling at  $90.0 \text{ km/h}$ . At that speed the forces resisting motion, including friction and air resistance, total  $400 \text{ N}$ . (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force the motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is  $245 \text{ kg}$ ?

**Exercise:**

**Problem:**

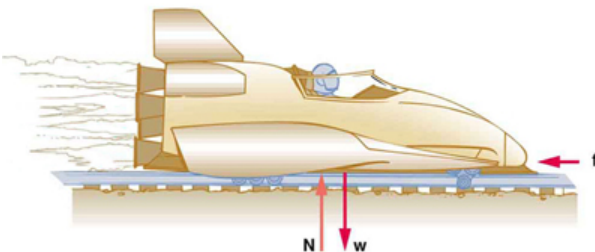
The rocket sled shown in [\[link\]](#) accelerates at a rate of  $49.0 \text{ m/s}^2$ . Its passenger has a mass of  $75.0 \text{ kg}$ . (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight by using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.

---

**Solution:**

(a)  $3.68 \times 10^3 \text{ N}$ . This force is 5.00 times greater than his weight.

(b)  $3750 \text{ N}$ ;  $11.3^\circ$  above horizontal



**Exercise:****Problem:**

Repeat the previous problem for the situation in which the rocket sled decelerates at a rate of  $201 \text{ m/s}^2$ . In this problem, the forces are exerted by the seat and restraining belts.

**Exercise:****Problem:**

The weight of an astronaut plus his space suit on the Moon is only 250 N. How much do they weigh on Earth? What is the mass on the Moon? On Earth?

---

**Solution:**

$1.5 \times 10^3 \text{ N}$ , 150 kg, 150 kg

**Exercise:****Problem:**

Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 10,000 kg. The thrust of its engines is 30,000 N. (a) Calculate its the magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.

**Glossary**

acceleration

the rate at which an object's velocity changes over a period of time

free-fall

a situation in which the only force acting on an object is the force due to gravity

friction



a force past each other of objects that are touching; examples include rough surfaces and air resistance

net external force

the vector sum of all external forces acting on an object or system; causes a mass to accelerate

Newton's second law of motion

the net external force  $\mathbf{F}_{\text{net}}$  on an object with mass  $m$  is proportional to and in the same direction as the acceleration of the object,  $\mathbf{a}$ , and inversely proportional to the mass; defined mathematically as

$$\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$$

system

defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces

weight

the force  $\mathbf{w}$  due to gravity acting on an object of mass  $m$ ; defined mathematically as:  $\mathbf{w} = m\mathbf{g}$ , where  $\mathbf{g}$  is the magnitude and direction of the acceleration due to gravity

## Newton's Third Law of Motion: Symmetry in Forces

- Understand Newton's third law of motion.
- Apply Newton's third law to define systems and solve problems of motion.

There is a passage in the musical *Man of la Mancha* that relates to Newton's third law of motion. Sancho, in describing a fight with his wife to Don Quixote, says, "Of course I hit her back, Your Grace, but she's a lot harder than me and you know what they say, 'Whether the stone hits the pitcher or the pitcher hits the stone, it's going to be bad for the pitcher.'" This is exactly what happens whenever one body exerts a force on another—the first also experiences a force (equal in magnitude and opposite in direction). Numerous common experiences, such as stubbing a toe or throwing a ball, confirm this. It is precisely stated in **Newton's third law of motion**.

### **Note:**

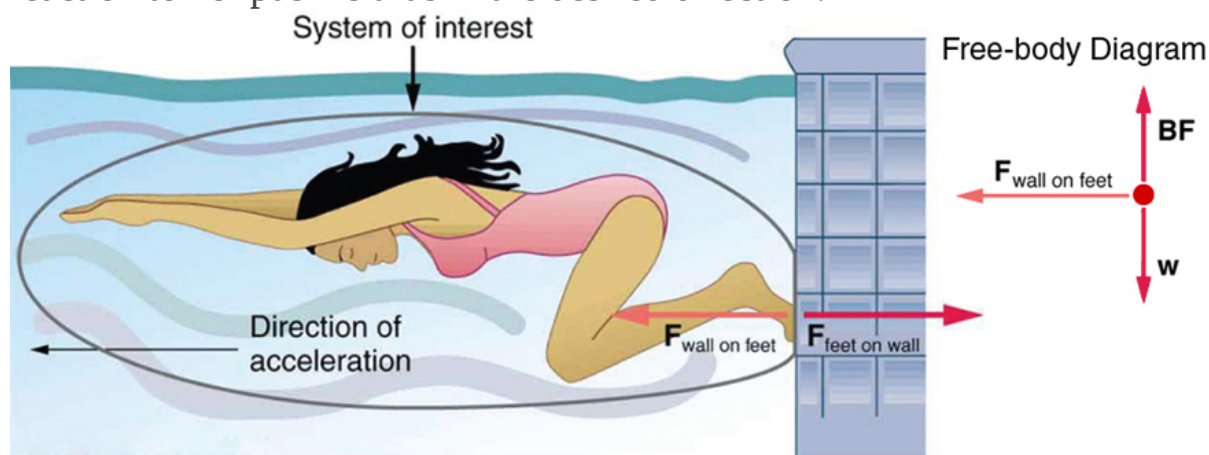
#### Newton's Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain *symmetry in nature*: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as "action-reaction," where the force exerted is the action and the force experienced as a consequence is the reaction. Newton's third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton's third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in [\[link\]](#). She pushes against the pool wall with her feet

and accelerates in the direction *opposite* to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not *because they act on different systems*. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then  $\mathbf{F}_{\text{wall on feet}}$  is an external force on this system and affects its motion. The swimmer moves in the direction of  $\mathbf{F}_{\text{wall on feet}}$ . In contrast, the force  $\mathbf{F}_{\text{feet on wall}}$  acts on the wall and not on our system of interest. Thus  $\mathbf{F}_{\text{feet on wall}}$  does not directly affect the motion of the system and does not cancel  $\mathbf{F}_{\text{wall on feet}}$ . Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.

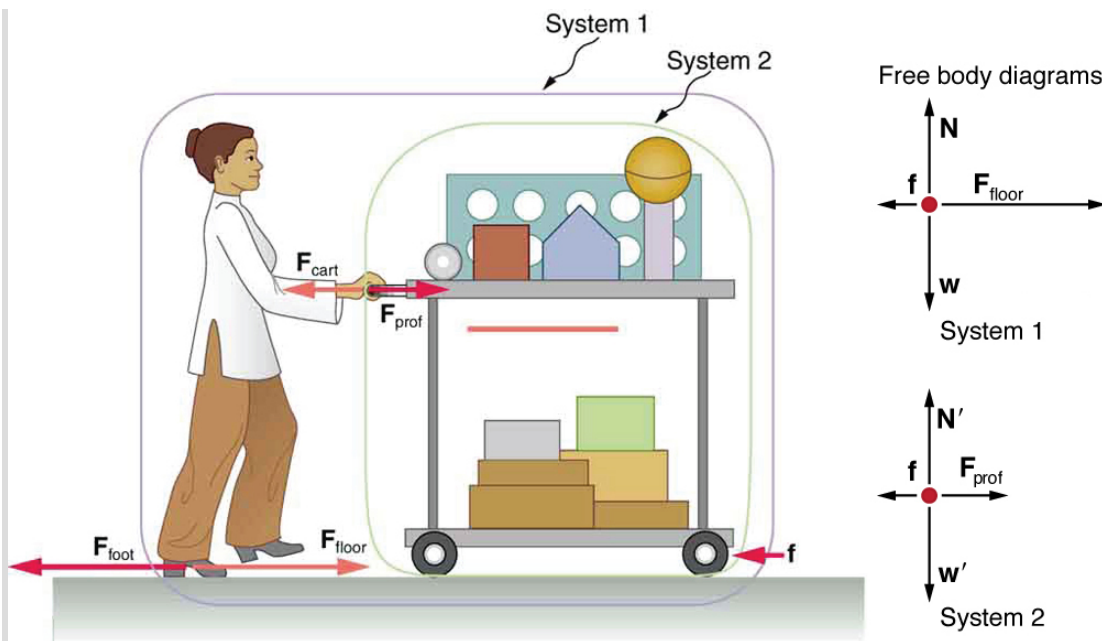


When the swimmer exerts a force  $\mathbf{F}_{\text{feet on wall}}$  on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to  $\mathbf{F}_{\text{feet on wall}}$ . This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force  $\mathbf{F}_{\text{wall on feet}}$  on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that  $\mathbf{F}_{\text{feet on wall}}$  does not act on this system (the swimmer) and, thus, does not cancel  $\mathbf{F}_{\text{wall on feet}}$ . Thus the free-body diagram shows only  $\mathbf{F}_{\text{wall on feet}}$ ,  $\mathbf{w}$ , the gravitational force, and  $\mathbf{BF}$ , the buoyant force of the water supporting the swimmer's weight. The vertical forces  $\mathbf{w}$  and  $\mathbf{BF}$  cancel since there is no vertical motion.

Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward. In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho's, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent's body.

**Example:****Getting Up To Speed: Choosing the Correct System**

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in [\[link\]](#). Her mass is 65.0 kg, the cart's is 12.0 kg, and the equipment's is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.



A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for  $f$ , since it is too small to draw to scale). Different questions are asked in each example; thus, the system of interest must be defined differently for each. System 1 is appropriate for this example, since it asks for the acceleration of the entire group of objects. Only  $F_{\text{floor}}$  and  $f$  are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for [\[link\]](#) so that  $F_{\text{prof}}$  will be an external force and enter into Newton's second law. Note that the free-body diagrams, which allow us to apply Newton's second law, vary with the system chosen.

### Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in [\[link\]](#). The professor pushes backward with a force  $F_{\text{foot}}$  of 150 N. According to Newton's third law, the floor exerts a forward reaction force  $F_{\text{floor}}$  of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the

horizontal direction. As noted,  $\mathbf{f}$  opposes the motion and is thus in the opposite direction of  $\mathbf{F}_{\text{floor}}$ . Note that we do not include the forces  $\mathbf{F}_{\text{prof}}$  or  $\mathbf{F}_{\text{cart}}$  because these are internal forces, and we do not include  $\mathbf{F}_{\text{foot}}$  because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

### **Solution**

Newton's second law is given by

**Equation:**

$$a = \frac{F_{\text{net}}}{m}.$$

The net external force on System 1 is deduced from [\[link\]](#) and the discussion above to be

**Equation:**

$$F_{\text{net}} = F_{\text{floor}} - f = 150 \text{ N} - 24.0 \text{ N} = 126 \text{ N}.$$

The mass of System 1 is

**Equation:**

$$m = (65.0 + 12.0 + 7.0) \text{ kg} = 84 \text{ kg}.$$

These values of  $F_{\text{net}}$  and  $m$  produce an acceleration of

**Equation:**

$$a = \frac{F_{\text{net}}}{m},$$
$$a = \frac{126 \text{ N}}{84 \text{ kg}} = 1.5 \text{ m/s}^2.$$

### **Discussion**

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the

professor on the cart results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

**Example:****Force on the Cart—Choosing a New System**

Calculate the force the professor exerts on the cart in [\[link\]](#) using data from the previous example if needed.

**Strategy**

If we now define the system of interest to be the cart plus equipment (System 2 in [\[link\]](#)), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart,  $\mathbf{F}_{\text{prof}}$ , is an external force acting on System 2.  $\mathbf{F}_{\text{prof}}$  was internal to System 1, but it is external to System 2 and will enter Newton's second law for System 2.

**Solution**

Newton's second law can be used to find  $\mathbf{F}_{\text{prof}}$ . Starting with

**Equation:**

$$a = \frac{F_{\text{net}}}{m}$$

and noting that the magnitude of the net external force on System 2 is

**Equation:**

$$F_{\text{net}} = F_{\text{prof}} - f,$$

we solve for  $F_{\text{prof}}$ , the desired quantity:

**Equation:**

$$F_{\text{prof}} = F_{\text{net}} + f.$$

The value of  $f$  is given, so we must calculate net  $F_{\text{net}}$ . That can be done since both the acceleration and mass of System 2 are known. Using Newton's second law we see that

**Equation:**

$$F_{\text{net}} = ma,$$

where the mass of System 2 is 19.0 kg ( $m = 12.0 \text{ kg} + 7.0 \text{ kg}$ ) and its acceleration was found to be  $a = 1.5 \text{ m/s}^2$  in the previous example. Thus,

**Equation:**

$$F_{\text{net}} = ma,$$

**Equation:**

$$F_{\text{net}} = (19.0 \text{ kg})(1.5 \text{ m/s}^2) = 29 \text{ N}.$$

Now we can find the desired force:

**Equation:**

$$F_{\text{prof}} = F_{\text{net}} + f,$$

**Equation:**

$$F_{\text{prof}} = 29 \text{ N} + 24.0 \text{ N} = 53 \text{ N}.$$

### Discussion

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

### Note:

PhET Explorations: Gravity Force Lab

Visualize the gravitational force that two objects exert on each other.

Change properties of the objects in order to see how it changes the gravity force.



[https://phet.colorado.edu/sims/html/gravity-force-lab/latest/gravity-force-lab\\_en.html](https://phet.colorado.edu/sims/html/gravity-force-lab/latest/gravity-force-lab_en.html)

## Section Summary

- **Newton's third law of motion** represents a basic symmetry in nature. It states: Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts.
- A **thrust** is a reaction force that pushes a body forward in response to a backward force. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.

## Conceptual Questions

### Exercise:

#### Problem:

When you take off in a jet aircraft, there is a sensation of being pushed back into the seat. Explain why you move backward in the seat—is there really a force backward on you? (The same reasoning explains whiplash injuries, in which the head is apparently thrown backward.)

### Exercise:

#### Problem:

A device used since the 1940s to measure the kick or recoil of the body due to heart beats is the “ballistocardiograph.” What physics principle(s) are involved here to measure the force of cardiac contraction? How might we construct such a device?

### Exercise:

**Problem:**

Describe a situation in which one system exerts a force on another and, as a consequence, experiences a force that is equal in magnitude and opposite in direction. Which of Newton's laws of motion apply?

**Exercise:****Problem:**

Why does an ordinary rifle recoil (kick backward) when fired? The barrel of a recoilless rifle is open at both ends. Describe how Newton's third law applies when one is fired. Can you safely stand close behind one when it is fired?

**Exercise:****Problem:**

An American football lineman reasons that it is senseless to try to out-push the opposing player, since no matter how hard he pushes he will experience an equal and opposite force from the other player. Use Newton's laws and draw a free-body diagram of an appropriate system to explain how he can still out-push the opposition if he is strong enough.

**Exercise:****Problem:**

Newton's third law of motion tells us that forces always occur in pairs of equal and opposite magnitude. Explain how the choice of the "system of interest" affects whether one such pair of forces cancels.

**Problem Exercises****Exercise:**

**Problem:**

What net external force is exerted on a 1100-kg artillery shell fired from a battleship if the shell is accelerated at  $2.40 \times 10^4 \text{ m/s}^2$ ? What is the magnitude of the force exerted on the ship by the artillery shell?

---

**Solution:**

Force on shell:  $2.64 \times 10^7 \text{ N}$

Force exerted on ship =  $-2.64 \times 10^7 \text{ N}$ , by Newton's third law

**Exercise:****Problem:**

A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating at  $1.20 \text{ m/s}^2$  backward. (a) What is the force of friction between the losing player's feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110 kg? (c) Draw a sketch of the situation showing the system of interest used to solve each part. For this situation, draw a free-body diagram and write the net force equation.

**Glossary****Newton's third law of motion**

whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts

**thrust**

a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force

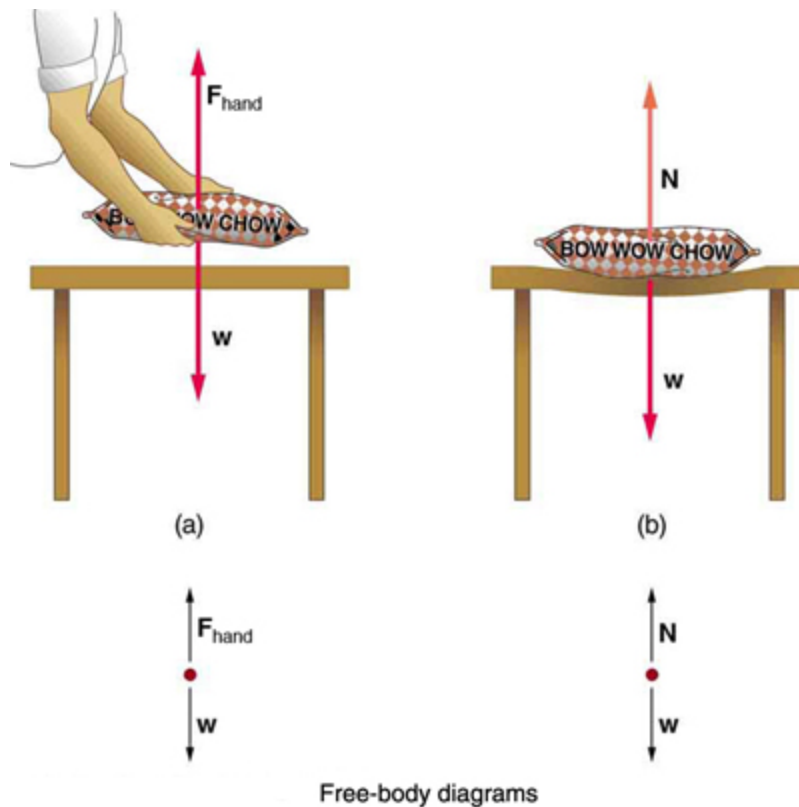
## Normal, Tension, and Other Examples of Forces

- Define normal and tension forces.
- Apply Newton's laws of motion to solve problems involving a variety of forces.
- Use trigonometric identities to resolve weight into components.

Forces are given many names, such as push, pull, thrust, lift, weight, friction, and tension. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. The most important of these categories are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this text.

### Normal Force

**Weight** (also called force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You definitely notice that you must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in [\[link\]](#)(a). But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in [\[link\]](#)(b)? When the bag of dog food is placed on the table, the table actually sags slightly under the load. This would be noticeable if the load were placed on a card table, but even rigid objects deform when a force is applied to them. Unless the object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or trampoline or diving board). The greater the deformation, the greater the restoring force. So when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly, and the sag is slight so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.



(a) The person holding the bag of dog food must supply an upward force  $\mathbf{F}_{\text{hand}}$  equal in magnitude and opposite in direction to the weight of the food  $\mathbf{w}$ . (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force  $\mathbf{N}$  equal in magnitude and opposite in direction to the weight of the load.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a **normal force** and here is given the symbol  $\mathbf{N}$ . (This is not the unit for force N.) The word *normal* means perpendicular to a

surface. The normal force can be less than the object's weight if the object is on an incline, as you will see in the next example.

**Note:**

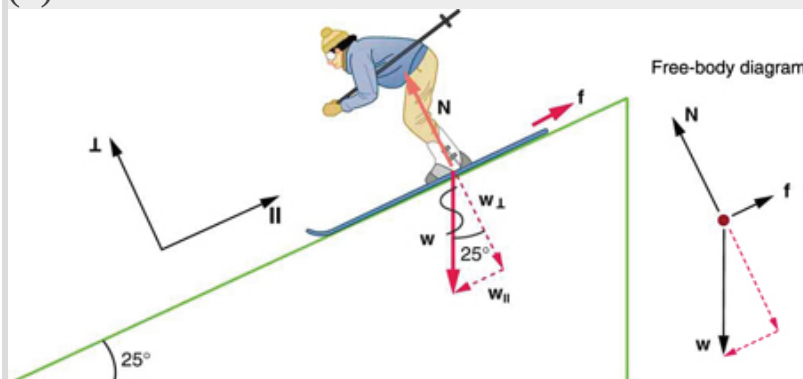
**Common Misconception: Normal Force ( $\mathbf{N}$ ) vs. Newton ( $\text{N}$ )**

In this section we have introduced the quantity normal force, which is represented by the variable  $\mathbf{N}$ . This should not be confused with the symbol for the newton, which is also represented by the letter  $\text{N}$ . These symbols are particularly important to distinguish because the units of a normal force ( $\mathbf{N}$ ) happen to be newtons ( $\text{N}$ ). For example, the normal force  $\mathbf{N}$  that the floor exerts on a chair might be  $\mathbf{N} = 100 \text{ N}$ . One important difference is that normal force is a vector, while the newton is simply a unit. Be careful not to confuse these letters in your calculations! You will encounter more similarities among variables and units as you proceed in physics. Another example of this is the quantity work ( $W$ ) and the unit watts ( $\text{W}$ ).

**Example:**

**Weight on an Incline, a Two-Dimensional Problem**

Consider the skier on a slope shown in [\[link\]](#). Her mass including equipment is  $60.0 \text{ kg}$ . (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is known to be  $45.0 \text{ N}$ ?



Since motion and friction are parallel to the slope, it is most convenient to project all

forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier).

$\mathbf{N}$  is perpendicular to the slope and  $\mathbf{f}$  is parallel to the slope, but  $\mathbf{w}$  has components along both axes, namely  $\mathbf{w}_\perp$  and  $\mathbf{w}_\parallel$ .  $\mathbf{N}$  is equal in magnitude to  $\mathbf{w}_\perp$ , so that there is no motion perpendicular to the slope, but  $f$  is less than  $w_\parallel$ , so that there is a downslope acceleration (along the parallel axis).

### Strategy

This is a two-dimensional problem, since the forces on the skier (the system of interest) are not parallel. The approach we have used in two-dimensional kinematics also works very well here. Choose a convenient coordinate system and project the vectors onto its axes, creating two connected *one*-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Remember that motions along mutually perpendicular axes are independent.) We use the symbols  $\perp$  and  $\parallel$  to represent perpendicular and parallel, respectively. This choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and because friction is always parallel to the surface between two objects. The only external forces acting on the system are the skier's weight, friction, and the support of the slope, respectively labeled  $\mathbf{w}$ ,  $\mathbf{f}$ , and  $\mathbf{N}$  in [\[link\]](#).  $\mathbf{N}$  is always perpendicular to the slope, and  $\mathbf{f}$  is parallel to it. But  $\mathbf{w}$  is not in the direction of either axis, and so the first step we take is to project it into components along the chosen axes, defining  $w_\parallel$  to be the component of weight parallel to the slope and  $w_\perp$  the component of weight perpendicular to the slope. Once this is done, we can consider the two separate problems of forces parallel to the slope and forces perpendicular to the slope.

### Solution

The magnitude of the component of the weight parallel to the slope is  $w_\parallel = w \sin(25^\circ) = mg \sin(25^\circ)$ , and the magnitude of the component of

the weight perpendicular to the slope is

$$w_{\perp} = w \cos (25^{\circ}) = mg \cos (25^{\circ}).$$

(a) Neglecting friction. Since the acceleration is parallel to the slope, we need only consider forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the amount of the skier's weight parallel to the slope  $w_{\parallel}$  and friction  $f$ . Using Newton's second law, with subscripts to denote quantities parallel to the slope,

**Equation:**

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$$

where  $F_{\text{net}\parallel} = w_{\parallel} = mg \sin (25^{\circ})$ , assuming no friction for this part, so that

**Equation:**

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{mg \sin (25^{\circ})}{m} = g \sin (25^{\circ})$$

**Equation:**

$$(9.80 \text{ m/s}^2)(0.4226) = 4.14 \text{ m/s}^2$$

is the acceleration.

(b) Including friction. We now have a given value for friction, and we know its direction is parallel to the slope and it opposes motion between surfaces in contact. So the net external force is now

**Equation:**

$$F_{\text{net}\parallel} = w_{\parallel} - f,$$

and substituting this into Newton's second law,  $a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$ , gives

**Equation:**

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{w_{\parallel} - f}{m} = \frac{mg \sin (25^{\circ}) - f}{m}.$$



We substitute known values to obtain

**Equation:**

$$a_{\parallel} = \frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(0.4226) - 45.0 \text{ N}}{60.0 \text{ kg}},$$

which yields

**Equation:**

$$a_{\parallel} = 3.39 \text{ m/s}^2,$$

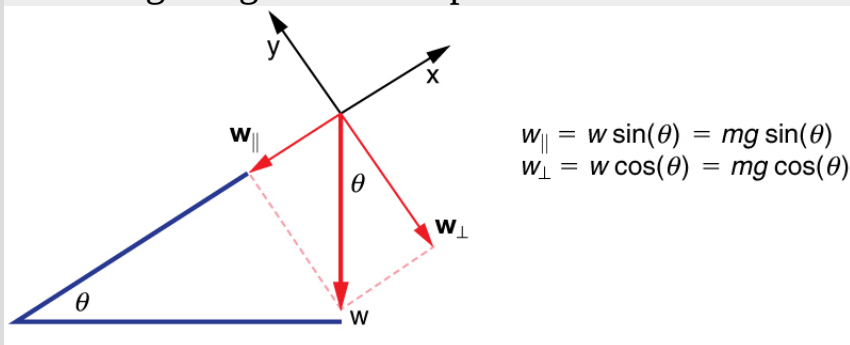
which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

### Discussion

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. In fact, it is a general result that if friction on an incline is negligible, then the acceleration down the incline is  $a = g \sin\theta$ , *regardless of mass*. This is related to the previously discussed fact that all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

### Note:

#### Resolving Weight into Components



An object rests on an incline that makes an

angle  $\theta$  with the horizontal.

When an object rests on an incline that makes an angle  $\theta$  with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane,  $\mathbf{w}_{\perp}$ , and a force acting parallel to the plane,  $\mathbf{w}_{\parallel}$ . The perpendicular force of weight,  $\mathbf{w}_{\perp}$ , is typically equal in magnitude and opposite in direction to the normal force,  $\mathbf{N}$ . The force acting parallel to the plane,  $\mathbf{w}_{\parallel}$ , causes the object to accelerate down the incline. The force of friction,  $\mathbf{f}$ , opposes the motion of the object, so it acts upward along the plane.

It is important to be careful when resolving the weight of the object into components. If the angle of the incline is at an angle  $\theta$  to the horizontal, then the magnitudes of the weight components are

**Equation:**

$$w_{\parallel} = w \sin (\theta) = mg \sin (\theta)$$

and

**Equation:**

$$w_{\perp} = w \cos (\theta) = mg \cos (\theta).$$

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, draw the right triangle formed by the three weight vectors. Notice that the angle  $\theta$  of the incline is the same as the angle formed between  $\mathbf{w}$  and  $\mathbf{w}_{\perp}$ . Knowing this property, you can use trigonometry to determine the magnitude of the weight components:

**Equation:**

$$\begin{aligned} \cos (\theta) &= \frac{w_{\perp}}{w} \\ w_{\perp} &= w \cos (\theta) = mg \cos (\theta) \end{aligned}$$

**Equation:**

$$\begin{aligned} \sin (\theta) &= \frac{w_{\parallel}}{w} \\ w_{\parallel} &= w \sin (\theta) = mg \sin (\theta) \end{aligned}$$

### **Note:**

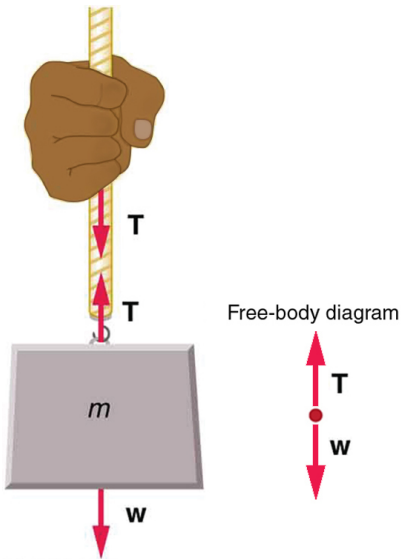
#### **Take-Home Experiment: Force Parallel**

To investigate how a force parallel to an inclined plane changes, find a rubber band, some objects to hang from the end of the rubber band, and a board you can position at different angles. How much does the rubber band stretch when you hang the object from the end of the board? Now place the board at an angle so that the object slides off when placed on the board. How much does the rubber band extend if it is lined up parallel to the board and used to hold the object stationary on the board? Try two more angles. What does this show?

## **Tension**

A **tension** is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word “tension” comes from a Latin word meaning “to stretch.” Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called *tendons*. Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: “You can’t push a rope.” The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in [\[link\]](#).



When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force  $\mathbf{T}$ , that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton's third law. The rope is the medium that carries

the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton's second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus  $\mathbf{F}_{\text{net}} = 0$ . The only external forces acting on the mass are its weight  $\mathbf{w}$  and the tension  $\mathbf{T}$  supplied by the rope. Thus,

**Equation:**

$$F_{\text{net}} = T - w = 0,$$

where  $T$  and  $w$  are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here. Thus, just as you would expect, the tension equals the weight of the supported mass:

**Equation:**

$$T = w = mg.$$

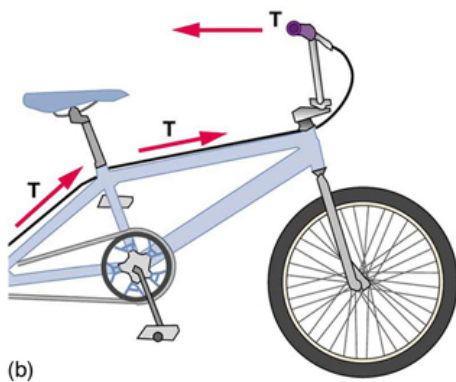
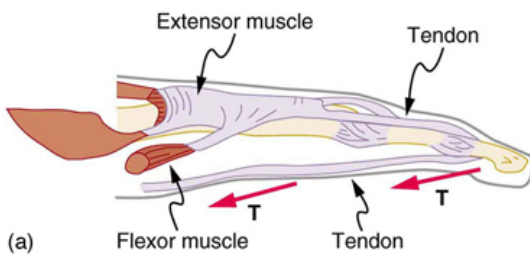
For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

### Equation:

$$T = mg = (5.00 \text{ kg})(9.80 \text{ m/s}^2) = 49.0 \text{ N}.$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction changes, and it is always parallel to the flexible connector. This is illustrated in [\[link\]](#) (a) and (b).



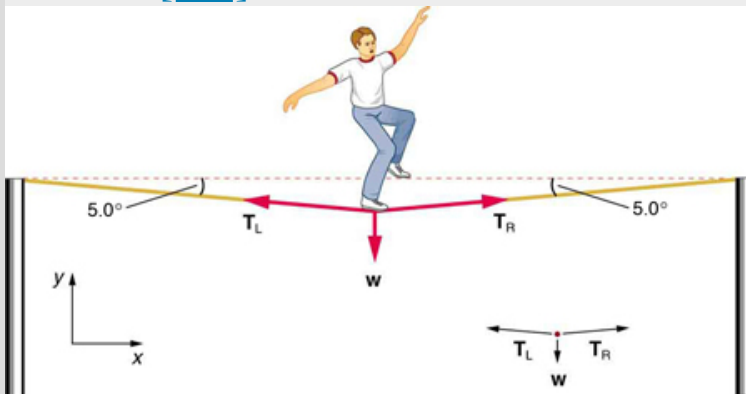
(a) Tendons in the finger carry force  $\mathbf{T}$  from the muscles to other parts of the finger, usually changing the force's direction, but not its magnitude (the

tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension  $\mathbf{T}$  from the handlebars to the brake mechanism. Again, the direction but not the magnitude of  $\mathbf{T}$  is changed.

### Example:

#### What Is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in [\[link\]](#).



The weight of a tightrope walker causes a wire to sag by 5.0 degrees. The system of interest here is the point in the wire at which the tightrope walker is standing.

### Strategy

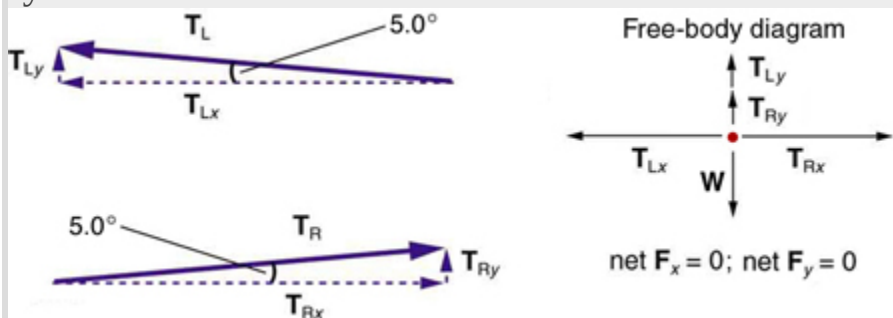
As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As

usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight  $\mathbf{w}$  and the two tensions  $\mathbf{T}_L$  (left tension) and  $\mathbf{T}_R$  (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions  $T_L$  and  $T_R$  must be equal. This is because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are  $T_L$  and  $T_R$ . Thus, the magnitude of those forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best coordinate system has one axis horizontal and the other vertical. We call the horizontal the  $x$ -axis and the vertical the  $y$ -axis.

### Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.



When the vectors are projected onto vertical and horizontal axes, their components along those axes must add to zero, since the tightrope walker is stationary. The small angle results in  $T$  being much greater than  $w$ .



Consider the horizontal components of the forces (denoted with a subscript  $x$ ):

**Equation:**

$$F_{\text{net}x} = T_{Lx} - T_{Rx}.$$

The net external horizontal force  $F_{\text{net}x} = 0$ , since the person is stationary. Thus,

**Equation:**

$$\begin{aligned} F_{\text{net}x} = 0 &= T_{Lx} - T_{Rx} \\ T_{Lx} &= T_{Rx}. \end{aligned}$$

Now, observe [\[link\]](#). You can use trigonometry to determine the magnitude of  $T_L$  and  $T_R$ . Notice that:

**Equation:**

$$\begin{aligned} \cos(5.0^\circ) &= \frac{T_{Lx}}{T_L} \\ T_{Lx} &= T_L \cos(5.0^\circ) \\ \cos(5.0^\circ) &= \frac{T_{Rx}}{T_R} \\ T_{Rx} &= T_R \cos(5.0^\circ). \end{aligned}$$

Equating  $T_{Lx}$  and  $T_{Rx}$ :

**Equation:**

$$T_L \cos(5.0^\circ) = T_R \cos(5.0^\circ).$$

Thus,

**Equation:**

$$T_L = T_R = T,$$

as predicted. Now, considering the vertical components (denoted by a subscript  $y$ ), we can solve for  $T$ . Again, since the person is stationary, Newton's second law implies that net  $F_y = 0$ . Thus, as illustrated in the free-body diagram in [\[link\]](#),

**Equation:**

$$F_{\text{net}y} = T_{Ly} + T_{Ry} - w = 0.$$

Observing [\[link\]](#), we can use trigonometry to determine the relationship between  $T_{Ly}$ ,  $T_{Ry}$ , and  $T$ . As we determined from the analysis in the horizontal direction,  $T_L = T_R = T$ :

**Equation:**

$$\begin{aligned}\sin (5.0^\circ) &= \frac{T_{Ly}}{T_L} \\ T_{Ly} = T_L \sin (5.0^\circ) &= T \sin (5.0^\circ) \\ \sin (5.0^\circ) &= \frac{T_{Ry}}{T_R} \\ T_{Ry} = T_R \sin (5.0^\circ) &= T \sin (5.0^\circ).\end{aligned}$$

Now, we can substitute the values for  $T_{Ly}$  and  $T_{Ry}$ , into the net force equation in the vertical direction:

**Equation:**

$$\begin{aligned}F_{\text{net}y} &= T_{Ly} + T_{Ry} - w = 0 \\ F_{\text{net}y} &= T \sin (5.0^\circ) + T \sin (5.0^\circ) - w = 0 \\ 2 T \sin (5.0^\circ) - w &= 0 \\ 2 T \sin (5.0^\circ) &= w\end{aligned}$$

and

**Equation:**

$$T = \frac{w}{2 \sin (5.0^\circ)} = \frac{mg}{2 \sin (5.0^\circ)},$$

so that

**Equation:**

$$T = \frac{(70.0 \text{ kg})(9.80 \text{ m/s}^2)}{2(0.0872)},$$

and the tension is

**Equation:**

$$T = 3900 \text{ N}.$$

### Discussion

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to *create* a very large tension, all we have to do is exert a force perpendicular to a flexible connector, as illustrated in [\[link\]](#). As we saw in the last example, the weight of the tightrope walker acted as a force perpendicular to the rope. We saw that the tension in the rope related to the weight of the tightrope walker in the following way:

**Equation:**

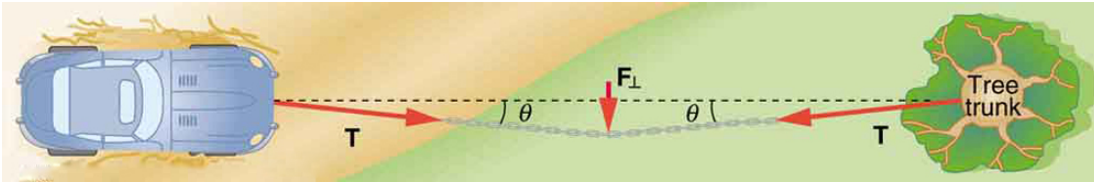
$$T = \frac{w}{2 \sin (\theta)}.$$

We can extend this expression to describe the tension  $T$  created when a perpendicular force ( $\mathbf{F}_{\perp}$ ) is exerted at the middle of a flexible connector:

**Equation:**

$$T = \frac{F_{\perp}}{2 \sin (\theta)}.$$

Note that  $\theta$  is the angle between the horizontal and the bent connector. In this case,  $T$  becomes very large as  $\theta$  approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e.,  $\theta = 0$  and  $\sin \theta = 0$ ). (See [\[link\]](#).)



We can create a very large tension in the chain by pushing on it perpendicular to its length, as shown. Suppose we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as nearly straight as possible. The tension in the chain is given by  $T = \frac{F_{\perp}}{2 \sin(\theta)}$ ; since  $\theta$  is small,  $T$  is very large. This situation is analogous to the tightrope walker shown in [\[link\]](#), except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where  $\mathbf{F}_{\perp}$  is applied.



Unless an infinite tension is exerted, any flexible connector—such as the chain at the bottom of the picture—will sag under its own weight, giving a characteristic curve when the weight is evenly

distributed along the length.

Suspension bridges—such as the Golden Gate Bridge shown in this image—are essentially very heavy flexible connectors. The weight of the bridge is evenly distributed along the length of flexible connectors, usually cables, which take on the characteristic shape.

(credit: Leaflet, Wikimedia Commons)

## Extended Topic: Real Forces and Inertial Frames

There is another distinction among forces in addition to the types already mentioned. Some forces are real, whereas others are not. *Real forces* are those that have some physical origin, such as the gravitational pull.

Contrastingly, *fictitious forces* are those that arise simply because an observer is in an accelerating frame of reference, such as one that rotates (like a merry-go-round) or undergoes linear acceleration (like a car slowing down). For example, if a satellite is heading due north above Earth's northern hemisphere, then to an observer on Earth it will appear to experience a force to the west that has no physical origin. Of course, what is happening here is that Earth is rotating toward the east and moves east under the satellite. In Earth's frame this looks like a westward force on the satellite, or it can be interpreted as a violation of Newton's first law (the law of inertia). An **inertial frame of reference** is one in which all forces are real and, equivalently, one in which Newton's laws have the simple forms given in this chapter.

Earth's rotation is slow enough that Earth is nearly an inertial frame. You ordinarily must perform precise experiments to observe fictitious forces and the slight departures from Newton's laws, such as the effect just described. On the large scale, such as for the rotation of weather systems and ocean currents, the effects can be easily observed.

The crucial factor in determining whether a frame of reference is inertial is whether it accelerates or rotates relative to a known inertial frame. Unless stated otherwise, all phenomena discussed in this text are considered in inertial frames.

All the forces discussed in this section are real forces, but there are a number of other real forces, such as lift and thrust, that are not discussed in this section. They are more specialized, and it is not necessary to discuss every type of force. It is natural, however, to ask where the basic simplicity we seek to find in physics is in the long list of forces. Are some more basic than others? Are some different manifestations of the same underlying force? The answer to both questions is yes, as will be seen in the next (extended) section and in the treatment of modern physics later in the text.

**Note:****PhET Explorations: Forces in 1 Dimension**

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. View a free-body diagram of all the forces (including gravitational and normal forces).

[Forces in  
1  
Dimension](#)

## Section Summary

- When objects rest on a surface, the surface applies a force to the object that supports the weight of the object. This supporting force acts

perpendicular to and away from the surface. It is called a normal force, **N**.

- When objects rest on a non-accelerating horizontal surface, the magnitude of the normal force is equal to the weight of the object:

**Equation:**

$$N = mg.$$

- When objects rest on an inclined plane that makes an angle  $\theta$  with the horizontal surface, the weight of the object can be resolved into components that act perpendicular ( $\mathbf{w}_\perp$ ) and parallel ( $\mathbf{w}_\parallel$ ) to the surface of the plane. These components can be calculated using:

**Equation:**

$$w_\parallel = w \sin(\theta) = mg \sin(\theta)$$

**Equation:**

$$w_\perp = w \cos(\theta) = mg \cos(\theta).$$

- The pulling force that acts along a stretched flexible connector, such as a rope or cable, is called tension, **T**. When a rope supports the weight of an object that is at rest, the tension in the rope is equal to the weight of the object:

**Equation:**

$$T = mg.$$

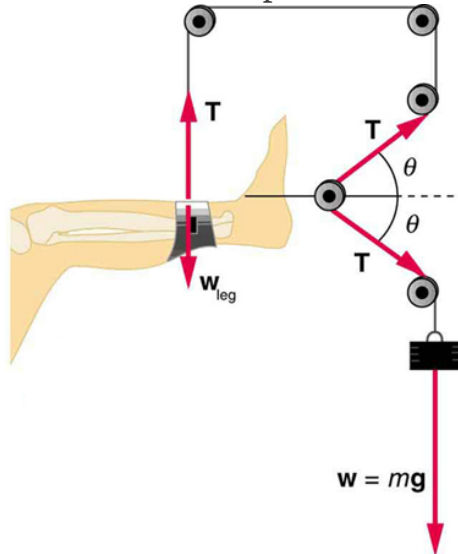
- In any inertial frame of reference (one that is not accelerated or rotated), Newton's laws have the simple forms given in this chapter and all forces are real forces having a physical origin.

## Conceptual Questions

**Exercise:**

**Problem:**

If a leg is suspended by a traction setup as shown in [\[link\]](#), what is the tension in the rope?



A leg is suspended by a traction system in which wires are used to transmit forces. Frictionless pulleys change the direction of the force  $T$  without changing its magnitude.

**Exercise:****Problem:**

In a traction setup for a broken bone, with pulleys and rope available, how might we be able to increase the force along the tibia using the same weight? (See [\[link\]](#).) (Note that the tibia is the shin bone shown in this image.)



## Problem Exercises

### Exercise:

#### Problem:

Two teams of nine members each engage in a tug of war. Each of the first team's members has an average mass of 68 kg and exerts an average force of 1350 N horizontally. Each of the second team's members has an average mass of 73 kg and exerts an average force of 1365 N horizontally. (a) What is magnitude of the acceleration of the two teams? (b) What is the tension in the section of rope between the teams?

---

#### Solution:

- a.  $0.11 \text{ m/s}^2$
- b.  $1.2 \times 10^4 \text{ N}$

### Exercise:

#### Problem:

What force does a trampoline have to apply to a 45.0-kg gymnast to accelerate her straight up at  $7.50 \text{ m/s}^2$ ? Note that the answer is independent of the velocity of the gymnast—she can be moving either up or down, or be stationary.

### Exercise:

#### Problem:

(a) Calculate the tension in a vertical strand of spider web if a spider of mass  $8.00 \times 10^{-5} \text{ kg}$  hangs motionless on it. (b) Calculate the tension in a horizontal strand of spider web if the same spider sits motionless in the middle of it much like the tightrope walker in [\[link\]](#). The strand sags at an angle of  $12^\circ$  below the horizontal. Compare this with the tension in the vertical strand (find their ratio).

---

**Solution:**

(a)  $7.84 \times 10^{-4} \text{ N}$

(b)  $1.89 \times 10^{-3} \text{ N}$  . This is 2.41 times the tension in the vertical strand.

**Exercise:****Problem:**

Suppose a 60.0-kg gymnast climbs a rope. (a) What is the tension in the rope if he climbs at a constant speed? (b) What is the tension in the rope if he accelerates upward at a rate of  $1.50 \text{ m/s}^2$ ?

**Exercise:****Problem:**

Show that, as stated in the text, a force  $\mathbf{F}_\perp$  exerted on a flexible medium at its center and perpendicular to its length (such as on the tightrope wire in [\[link\]](#)) gives rise to a tension of magnitude

$$T = \frac{F_\perp}{2 \sin(\theta)}.$$

---

**Solution:**

Newton's second law applied in vertical direction gives

**Equation:**

$$F_y = F - 2T \sin \theta = 0$$

**Equation:**

$$F = 2T \sin \theta$$

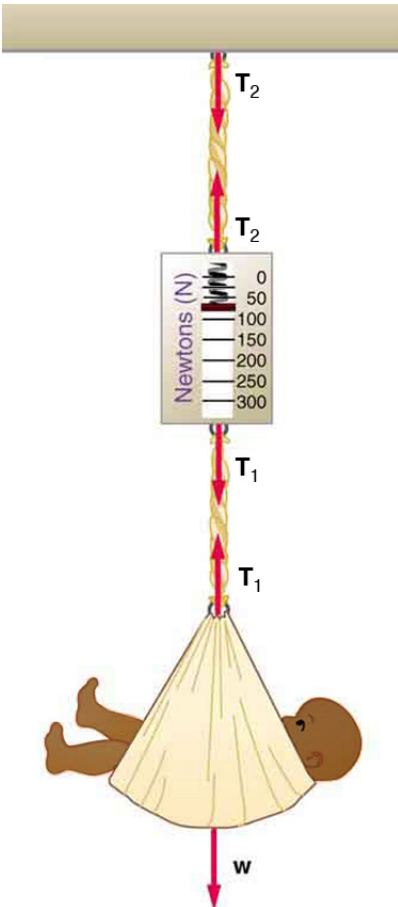
**Equation:**

$$T = \frac{F}{2 \sin \theta}.$$

### Exercise:

#### Problem:

Consider the baby being weighed in [\[link\]](#). (a) What is the mass of the child and basket if a scale reading of 55 N is observed? (b) What is the tension  $T_1$  in the cord attaching the baby to the scale? (c) What is the tension  $T_2$  in the cord attaching the scale to the ceiling, if the scale has a mass of 0.500 kg? (d) Draw a sketch of the situation indicating the system of interest used to solve each part. The masses of the cords are negligible.



A baby is weighed  
using a spring  
scale.

## Glossary

inertial frame of reference

a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference

normal force

the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests

tension

the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

## Friction

- Discuss the general characteristics of friction.
- Describe the various types of friction.
- Calculate the magnitude of static and kinetic friction.

**Friction** is a force that is around us all the time that opposes relative motion between systems in contact but also allows us to move (which you have discovered if you have ever tried to walk on ice). While a common force, the behavior of friction is actually very complicated and is still not completely understood. We have to rely heavily on observations for whatever understandings we can gain. However, we can still deal with its more elementary general characteristics and understand the circumstances in which it behaves.

### **Note:**

#### Friction

Friction is a force that opposes relative motion between systems in contact.

One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, **static friction** can act between them; the static friction is usually greater than the kinetic friction between the objects.

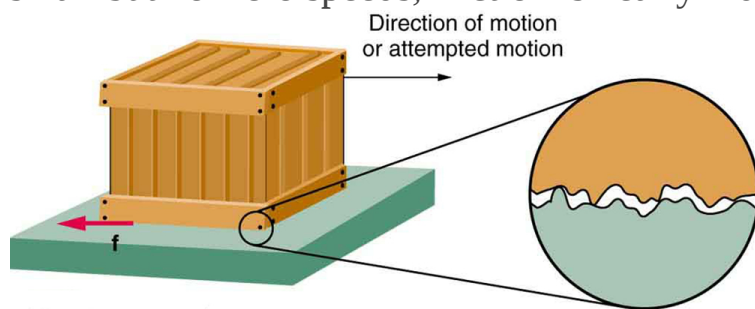
### **Note:**

#### Kinetic Friction

If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you may push harder and harder on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. But if you finally push hard enough, the crate seems to slip suddenly and starts to move. Once in motion it is easier to keep it in motion than it was to get it started, indicating that the kinetic friction force is less than the static friction force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it to be easier to get the crate started and keep it going (as you might expect).

[\[link\]](#) is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. So when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, break off the points, or do both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explain the dependence of friction on the nature of the substances. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.



Frictional forces, such as  $f$ , always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of

the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for when there is motion (kinetic friction).

When there is no motion between the objects, the **magnitude of static friction**  $f_s$  is

**Equation:**

$$f_s \leq \mu_s N,$$

where  $\mu_s$  is the coefficient of static friction and  $N$  is the magnitude of the normal force (the force perpendicular to the surface).

**Note:**

Magnitude of Static Friction

Magnitude of static friction  $f_s$  is

**Equation:**

$$f_s \leq \mu_s N,$$

where  $\mu_s$  is the coefficient of static friction and  $N$  is the magnitude of the normal force.

The symbol  $\leq$  means *less than or equal to*, implying that static friction can have a minimum and a maximum value of  $\mu_s N$ . Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds  $f_{s(\max)}$ , the object will move. Thus

**Equation:**

$$f_{s(\max)} = \mu_s N.$$

Once an object is moving, the **magnitude of kinetic friction**  $f_k$  is given by **Equation:**

$$f_k = \mu_k N,$$

where  $\mu_k$  is the coefficient of kinetic friction. A system in which  $f_k = \mu_k N$  is described as a system in which *friction behaves simply*.

**Note:****Magnitude of Kinetic Friction**

The magnitude of kinetic friction  $f_k$  is given by

**Equation:**

$$f_k = \mu_k N,$$

where  $\mu_k$  is the coefficient of kinetic friction.



As seen in [\[link\]](#), the coefficients of kinetic friction are less than their static counterparts. That values of  $\mu$  in [\[link\]](#) are stated to only one or, at most, two digits is an indication of the approximate description of friction given by the above two equations.

<b>System</b>	<b>Static friction</b> $\mu_s$	<b>Kinetic friction</b> $\mu_k$
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7

System	Static friction $\mu_s$	Kinetic friction $\mu_k$
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.04	0.02

### Coefficients of Static and Kinetic Friction

The equations given earlier include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force would be equal to its weight,  $W = mg = (100 \text{ kg})(9.80 \text{ m/s}^2) = 980 \text{ N}$ , perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than  $f_{s(\text{max})} = \mu_s N = (0.45)(980 \text{ N}) = 440 \text{ N}$  to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only 290 N ( $f_k = \mu_k N = (0.30)(980 \text{ N}) = 290 \text{ N}$ ) would keep it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unit less quantity with a magnitude usually between 0 and 1.0. The coefficient of the friction depends on the two surfaces that are in contact.

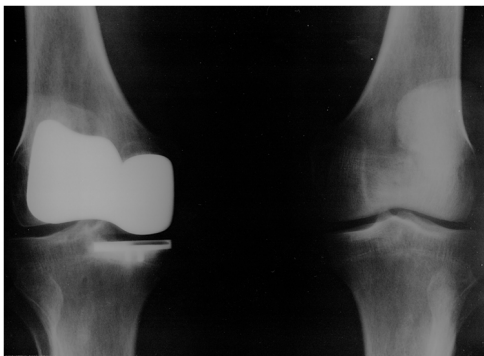
#### Note:

##### Take-Home Experiment

Find a small plastic object (such as a food container) and slide it on a kitchen table by giving it a gentle tap. Now spray water on the table,

simulating a light shower of rain. What happens now when you give the object the same-sized tap? Now add a few drops of (vegetable or olive) oil on the surface of the water and give the same tap. What happens now? This latter situation is particularly important for drivers to note, especially after a light rain shower. Why?

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint ([link](#)). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.



Artificial knee replacement is a procedure that has been performed for more than 20 years. In this figure, we see the post-op x rays of the right knee joint replacement. (credit: Mike Baird, Flickr)

Other natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Artificial lubricants are also common in hospitals and doctor's clinics. For example, when ultrasonic imaging is carried out, the gel that couples the transducer to the skin also serves to lubricate the surface between the transducer and the skin—thereby reducing the coefficient of friction between the two surfaces. This allows the transducer to move freely over the skin.

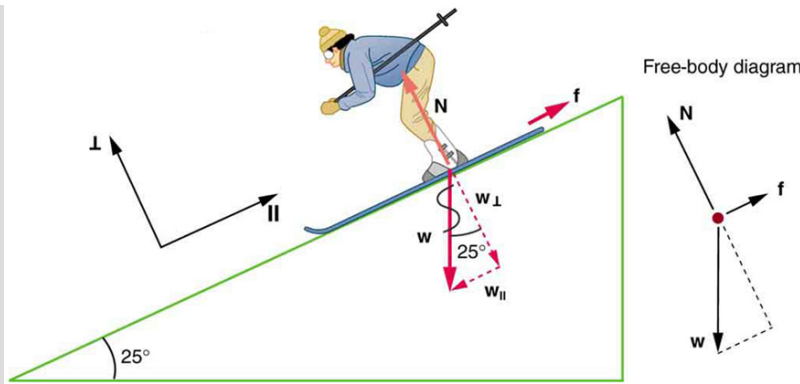
### **Example:**

#### **Skiing Exercise**

A skier with a mass of 62 kg is sliding down a snowy slope. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

#### **Strategy**

The magnitude of kinetic friction was given in to be 45.0 N. Kinetic friction is related to the normal force  $N$  as  $f_k = \mu_k N$ ; thus, the coefficient of kinetic friction can be found if we can find the normal force of the skier on a slope. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier's weight perpendicular to the slope. (See the skier and free-body diagram in [\[link\]](#).)



The motion of the skier and friction are parallel to the slope and so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier).  $\mathbf{N}$  (the normal force) is perpendicular to the slope, and  $\mathbf{f}$  (the friction) is parallel to the slope, but  $\mathbf{w}$  (the skier's weight) has components along both axes, namely  $\mathbf{w}_{\perp}$  and  $\mathbf{W}_{\parallel}$ .  $\mathbf{N}$  is equal in magnitude to  $\mathbf{w}_{\perp}$ , so there is no motion perpendicular to the slope. However,  $\mathbf{f}$  is less than  $\mathbf{W}_{\parallel}$  in magnitude, so there is acceleration down the slope (along the  $x$ -axis).

That is,

**Equation:**

$$N = w_{\perp} = w \cos 25^{\circ} = mg \cos 25^{\circ}.$$

Substituting this into our expression for kinetic friction, we get

**Equation:**

$$f_k = \mu_k mg \cos 25^{\circ},$$

which can now be solved for the coefficient of kinetic friction  $\mu_k$ .

**Solution**

Solving for  $\mu_k$  gives

**Equation:**

$$\mu_k = \frac{f_k}{N} = \frac{f_k}{w \cos 25^\circ} = \frac{f_k}{mg \cos 25^\circ}.$$

Substituting known values on the right-hand side of the equation,

**Equation:**

$$\mu_k = \frac{45.0 \text{ N}}{(62 \text{ kg})(9.80 \text{ m/s}^2)(0.906)} = 0.082.$$

**Discussion**

This result is a little smaller than the coefficient listed in [\[link\]](#) for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass  $m$  slides down a slope that makes an angle  $\theta$  with the horizontal, friction is given by  $f_k = \mu_k mg \cos \theta$ . All objects will slide down a slope with constant acceleration under these circumstances. Proof of this is left for this chapter's Problems and Exercises.

**Note:****Take-Home Experiment**

An object will slide down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in [\[link\]](#), the kinetic friction on a slope  $f_k = \mu_k mg \cos \theta$ . The component of the weight down the slope is equal to  $mg \sin \theta$  (see the free-body diagram in [\[link\]](#)). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out:

**Equation:**

$$f_k = F_{g_x}$$

**Equation:**

$$\mu_k mg \cos \theta = mg \sin \theta.$$

Solving for  $\mu_k$ , we find that

**Equation:**

$$\mu_k = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta.$$

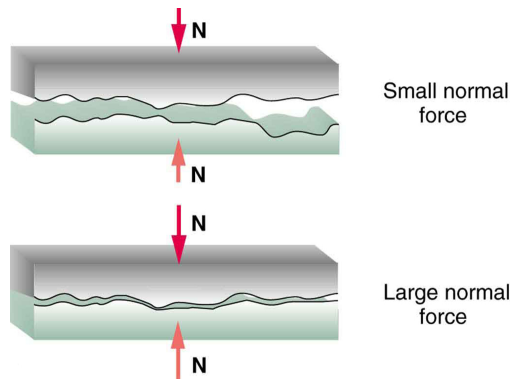
Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find  $\mu_k$ . Note that the coin will not start to slide at all until an angle greater than  $\theta$  is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Discuss how this may affect the value for  $\mu_k$  and its uncertainty.

We have discussed that when an object rests on a horizontal surface, there is a normal force supporting it equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force.

**Note:****Making Connections: Submicroscopic Explanations of Friction**

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) to heat.

[\[link\]](#) illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area since only high spots touch. When a greater normal force is exerted, the actual contact area increases, and it is found that the friction is proportional to this area.

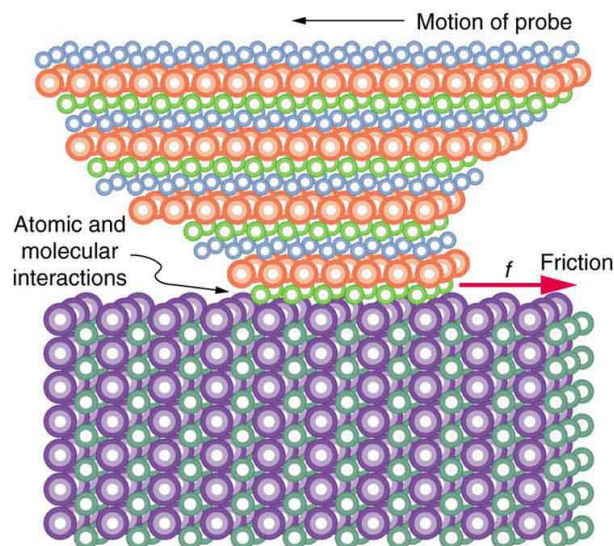


Two rough surfaces in contact have a much smaller area of actual contact than their total area. When there is a greater normal force as a result of a greater applied force, the area of actual contact increases as does friction.

But the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur



between atoms and molecules on the surfaces. [\[link\]](#) shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which will be discussed later in this chapter. The variation in shear stress is remarkable (more than a factor of  $10^{12}$ ) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.



The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

**Note:**

PhET Explorations: Forces and Motion

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. Draw a free-body diagram of all the forces (including gravitational and normal forces).

[Forces](#)  
[and](#)  
[Motion](#)  
[Simulation](#)

## Section Summary

- Friction is a contact force between systems that opposes the motion or attempted motion between them. Simple friction is proportional to the normal force  $N$  pushing the systems together. (A normal force is always perpendicular to the contact surface between systems.) Friction depends on both of the materials involved. The magnitude of static friction  $f_s$  between systems stationary relative to one another is given by

**Equation:**

$$f_s \leq \mu_s N,$$

where  $\mu_s$  is the coefficient of static friction, which depends on both of the materials.

- The kinetic friction force  $f_k$  between systems moving relative to one another is given by

**Equation:**

$$f_k = \mu_k N,$$

where  $\mu_k$  is the coefficient of kinetic friction, which also depends on both materials.

## Conceptual Questions

### Exercise:

#### Problem:

Define normal force. What is its relationship to friction when friction behaves simply?

### Exercise:

#### Problem:

The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.

### Exercise:

#### Problem:

When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.

### Exercise:

#### Problem:

When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)

## Problems & Exercises

### Exercise:

#### Problem:

A physics major is cooking breakfast when he notices that the frictional force between his steel spatula and his Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, he quickly calculates the normal force. What is it?

---

#### Solution:

5.00 N

### Exercise:

#### Problem:

(a) When rebuilding her car's engine, a physics major must exert 300 N of force to insert a dry steel piston into a steel cylinder. What is the magnitude of the normal force between the piston and cylinder? (b) What is the magnitude of the force would she have to exert if the steel parts were oiled?

### Exercise:

#### Problem:

(a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise it is possible to exert forces to the joints that are easily ten times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.

### Exercise:

**Problem:**

Suppose you have a 120-kg wooden crate resting on a wood floor. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will the magnitude of its acceleration then be?

---

**Solution:**

(a) 588 N

(b)  $1.96 \text{ m/s}^2$

**Exercise:****Problem:**

(a) If half of the weight of a small  $1.00 \times 10^3 \text{ kg}$  utility truck is supported by its two drive wheels, what is the magnitude of the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.

**Exercise:****Problem:**

A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the magnitude of the acceleration starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) What is the magnitude of the acceleration once the sled starts to move? (c) For both situations, calculate the magnitude of the force in the coupling between the dogs and the sled.

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**Solution:**

(a)  $3.29 \text{ m/s}^2$

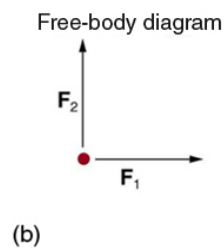
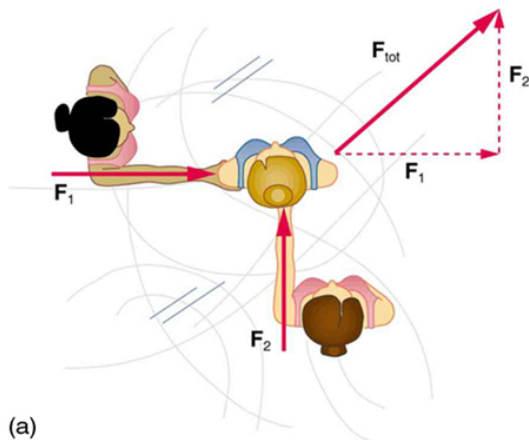
(b)  $3.52 \text{ m/s}^2$

(c) 980 N; 945 N

**Exercise:**

**Problem:**

Consider the 65.0-kg ice skater being pushed by two others shown in [\[link\]](#). (a) Find the direction and magnitude of  $\mathbf{F}_{\text{tot}}$ , the total force exerted on her by the others, given that the magnitudes  $F_1$  and  $F_2$  are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of  $\mathbf{F}_{\text{tot}}$ ? (c) What is her acceleration assuming she is already moving in the direction of  $\mathbf{F}_{\text{tot}}$ ? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)



**Exercise:**

**Problem:**

Show that the acceleration of any object down a frictionless incline that makes an angle  $\theta$  with the horizontal is  $a = g \sin \theta$ . (Note that this acceleration is independent of mass.)

**Exercise:**

**Problem:**

Show that the acceleration of any object down an incline where friction behaves simply (that is, where  $f_k = \mu_k N$ ) is  $a = g(\sin \theta - \mu_k \cos \theta)$ . Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small ( $\mu_k = 0$ ).

**Exercise:****Problem:**

Calculate the deceleration of a snow boarder going up a  $5.0^\circ$  slope assuming the coefficient of friction for waxed wood on wet snow. The result of [\[link\]](#) may be useful, but be careful to consider the fact that the snow boarder is going uphill. Explicitly show how you follow the steps in [Problem-Solving Strategies](#).

---

**Solution:**

$$1.83 \text{ m/s}^2$$

**Exercise:****Problem:**

(a) Calculate the acceleration of a skier heading down a  $10.0^\circ$  slope, assuming the coefficient of friction for waxed wood on wet snow. (b) Find the angle of the slope down which this skier could coast at a constant velocity. You can neglect air resistance in both parts, and you will find the result of [\[link\]](#) to be useful. Explicitly show how you follow the steps in the [Problem-Solving Strategies](#).

**Exercise:**

**Problem:**

If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is  $\theta = \tan^{-1} \mu_s$ . You may use the result of the previous problem. Assume that  $a = 0$  and that static friction has reached its maximum value.

**Exercise:****Problem:**

Calculate the maximum deceleration of a car that is heading down a  $6^\circ$  slope (one that makes an angle of  $6^\circ$  with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that  $\mu_s = 0.100$ , the same as for shoes on ice.

**Exercise:****Problem:**

Calculate the maximum acceleration of a car that is heading up a  $4^\circ$  slope (one that makes an angle of  $4^\circ$  with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that  $\mu_s = 0.100$ , the same as for shoes on ice.

---

**Solution:**

(a)  $4.20 \text{ m/s}^2$



(b)  $2.74 \text{ m/s}^2$

(c)  $-0.195 \text{ m/s}^2$

**Exercise:**

**Problem:** Repeat [\[link\]](#) for a car with four-wheel drive.

**Exercise:**

**Problem:**

A freight train consists of two  $8.00 \times 10^5$ -kg engines and 45 cars with average masses of  $5.50 \times 10^5$  kg. (a) What force must each engine exert backward on the track to accelerate the train at a rate of  $5.00 \times 10^{-2} \text{ m/s}^2$  if the force of friction is  $7.50 \times 10^5 \text{ N}$ , assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the magnitude of the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

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**Solution:**

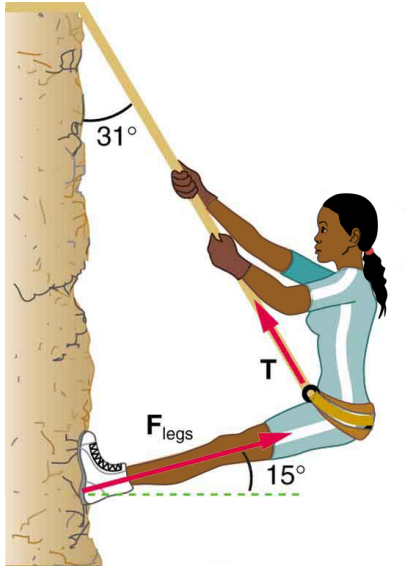
(a)  $1.03 \times 10^6 \text{ N}$

(b)  $3.48 \times 10^5 \text{ N}$

**Exercise:**

**Problem:**

Consider the 52.0-kg mountain climber in [\[link\]](#). (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?



Part of the climber's weight is supported by her rope and part by friction between her feet and the rock face.

### Exercise:

#### Problem:

A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown in [\[link\]](#)(a). (a) Calculate the minimum force  $F$  he must exert to get the block moving. (b) What is the magnitude of its acceleration once it starts to move, if that force is maintained?

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#### Solution:

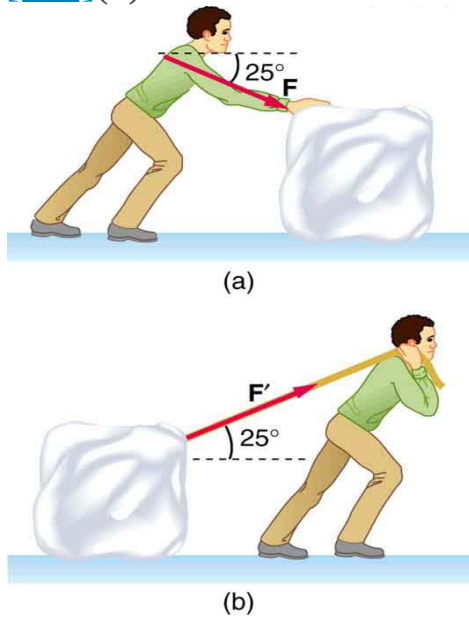
(a) 51.0 N

(b)  $0.720 \text{ m/s}^2$

## Exercise:

### Problem:

Repeat [\[link\]](#) with the contestant pulling the block of ice with a rope over his shoulder at the same angle above the horizontal as shown in [\[link\]](#)(b).



Which method of sliding a block of ice requires less force—(a) pushing or (b) pulling at the same angle above the horizontal?

## Glossary

### friction

a force that opposes relative motion or attempts at motion between systems in contact

### kinetic friction

a force that opposes the motion of two systems that are in contact and moving relative to one another

static friction

a force that opposes the motion of two systems that are in contact and are not moving relative to one another

magnitude of static friction

$f_s \leq \mu_s N$ , where  $\mu_s$  is the coefficient of static friction and  $N$  is the magnitude of the normal force

magnitude of kinetic friction

$f_k = \mu_k N$ , where  $\mu_k$  is the coefficient of kinetic friction

## Elasticity: Stress and Strain

- State Hooke's law.
- Explain Hooke's law using graphical representation between deformation and applied force.
- Discuss the three types of deformations such as changes in length, sideways shear and changes in volume.
- Describe with examples the young's modulus, shear modulus and bulk modulus.
- Determine the change in length given mass, length and radius.

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. A change in shape due to the application of a force is a **deformation**. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the force—that is, for small deformations, Hooke's law is obeyed. In equation form, **Hooke's law** is given by

**Equation:**

$$F = k\Delta L,$$

where  $\Delta L$  is the amount of deformation (the change in length, for example) produced by the force  $F$ , and  $k$  is a proportionality constant that depends on the shape and composition of the object and the direction of the force. Note that this force is a function of the deformation  $\Delta L$ —it is not constant as a kinetic friction force is. Rearranging this to

**Equation:**

$$\Delta L = \frac{F}{k}$$

makes it clear that the deformation is proportional to the applied force. [\[link\]](#) shows the Hooke's law relationship between the extension  $\Delta L$  of a spring or of a human bone. For metals or springs, the straight line region in which Hooke's law pertains is much larger. Bones are brittle and the elastic region is small and the fracture abrupt. Eventually a large enough stress to the material will cause it to break or fracture. **Tensile strength** is the breaking stress that will cause permanent deformation or fracture of a material.

**Note:**

Hooke's Law

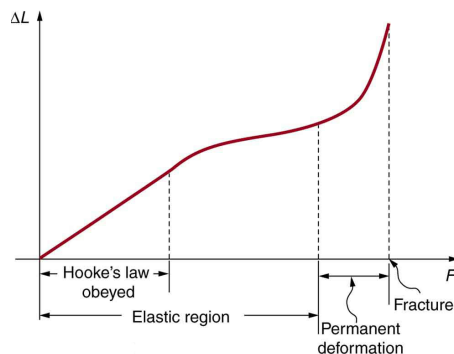
**Equation:**

$$F = k\Delta L,$$

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**Equation:**

$$\Delta L = \frac{F}{k}$$

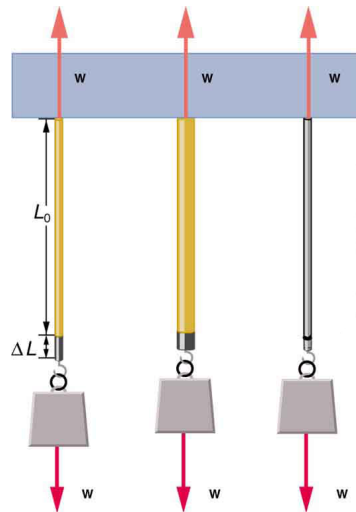


A graph of  
deformation  $\Delta L$   
versus applied force  $F$

. The straight segment is the linear region where Hooke's law is obeyed. The slope of the straight region is  $\frac{1}{k}$ . For larger forces, the graph is curved but the deformation is still elastic—  $\Delta L$  will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force  $F$  is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in  $F$  is producing a large increase in  $L$  near the fracture.

The proportionality constant  $k$  depends upon a number of factors for the material. For example, a guitar string made of nylon stretches when it is tightened, and the elongation  $\Delta L$  is proportional to the force applied (at least for small deformations). Thicker nylon strings and ones made of steel stretch less for the same applied force, implying they have a larger  $k$  (see [\[link\]](#)). Finally, all three strings return to their normal lengths when the force

is removed, provided the deformation is small. Most materials will behave in this manner if the deformation is less than about 0.1% or about 1 part in  $10^3$ .



The same force, in this case a weight ( $w$ ), applied to three different guitar strings of identical length produces the three different deformations shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.



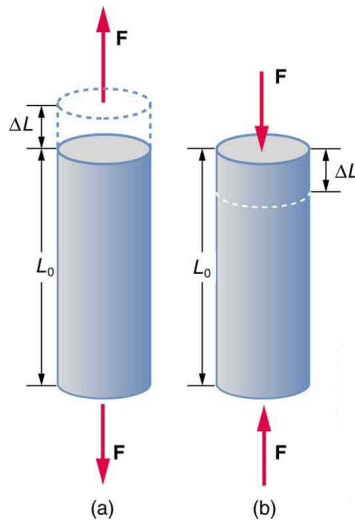
**Note:****Stretch Yourself a Little**

How would you go about measuring the proportionality constant  $k$  of a rubber band? If a rubber band stretched 3 cm when a 100-g mass was attached to it, then how much would it stretch if two similar rubber bands were attached to the same mass—even if put together in parallel or alternatively if tied together in series?

We now consider three specific types of deformations: changes in length (tension and compression), sideways shear (stress), and changes in volume. All deformations are assumed to be small unless otherwise stated.

**Changes in Length—Tension and Compression: Elastic Modulus**

A change in length  $\Delta L$  is produced when a force is applied to a wire or rod parallel to its length  $L_0$ , either stretching it (a tension) or compressing it. (See [\[link\]](#).)



(a) Tension. The rod is stretched

a length  $\Delta L$   
when a force is  
applied parallel  
to its length. (b)

Compression.  
The same rod is  
compressed by  
forces with the  
same magnitude  
in the opposite  
direction. For  
very small  
deformations  
and uniform  
materials,  $\Delta L$  is  
approximately  
the same for the  
same magnitude  
of tension or  
compression.

For larger  
deformations,  
the cross-  
sectional area  
changes as the  
rod is  
compressed or  
stretched.

Experiments have shown that the change in length ( $\Delta L$ ) depends on only a few variables. As already noted,  $\Delta L$  is proportional to the force  $F$  and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length  $L_0$  and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will

stretch less than a thin one. We can combine all these factors into one equation for  $\Delta L$ :

**Equation:**

$$\Delta L = \frac{1}{Y} \frac{F}{A} L_0,$$

where  $\Delta L$  is the change in length,  $F$  the applied force,  $Y$  is a factor, called the elastic modulus or Young’s modulus, that depends on the substance,  $A$  is the cross-sectional area, and  $L_0$  is the original length. [\[link\]](#) lists values of  $Y$  for several materials—those with a large  $Y$  are said to have a large tensile stiffness because they deform less for a given tension or compression.

Material	Young’s modulus (tension–compression) $Y$ ( $10^9$ N/m <sup>2</sup> )	Shear modulus $S$ ( $10^9$ N/m <sup>2</sup> )	Bulk modulus $B$ ( $10^9$ N/m <sup>2</sup> )
Aluminum	70	25	75
Bone – tension	16	80	8
Bone – compression	9		
Brass	90	35	75
Brick	15		

<b>Material</b>	<b>Young's modulus (tension–compression) <math>Y</math> (<math>10^9 \text{ N/m}^2</math>)</b>	<b>Shear modulus <math>S</math> (<math>10^9 \text{ N/m}^2</math>)</b>	<b>Bulk modulus <math>B</math> (<math>10^9 \text{ N/m}^2</math>)</b>
Concrete	20		
Glass	70	20	30
Granite	45	20	45
Hair (human)	10		
Hardwood	15	10	
Iron, cast	100	40	90
Lead	16	5	50
Marble	60	20	70
Nylon	5		
Polystyrene	3		
Silk	6		
Spider thread	3		
Steel	210	80	130
Tendon	1		

<b>Material</b>	<b>Young's modulus (tension–compression) <math>Y</math> (<math>10^9 \text{ N/m}^2</math>)</b>	<b>Shear modulus <math>S</math> (<math>10^9 \text{ N/m}^2</math>)</b>	<b>Bulk modulus <math>B</math> (<math>10^9 \text{ N/m}^2</math>)</b>
Acetone			0.7
Ethanol			0.9
Glycerin			4.5
Mercury			25
Water			2.2

Elastic Moduli[\[footnote\]](#)

Approximate and average values. Young's moduli  $Y$  for tension and compression sometimes differ but are averaged here. Bone has significantly different Young's moduli for tension and compression.

Young's moduli are not listed for liquids and gases in [\[link\]](#) because they cannot be stretched or compressed in only one direction. Note that there is an assumption that the object does not accelerate, so that there are actually two applied forces of magnitude  $F$  acting in opposite directions. For example, the strings in [\[link\]](#) are being pulled down by a force of magnitude  $w$  and held up by the ceiling, which also exerts a force of magnitude  $w$ .

### **Example:**

#### **The Stretch of a Long Cable**

Suspension cables are used to carry gondolas at ski resorts. (See [\[link\]](#)) Consider a suspension cable that includes an unsupported span of 3020 m. Calculate the amount of stretch in the steel cable. Assume that the cable has a diameter of 5.6 cm and the maximum tension it can withstand is  $3.0 \times 10^6 \text{ N}$ .



Gondolas travel along suspension cables at the Gala Yuzawa ski resort in Japan. (credit: Rudy Herman, Flickr)

### Strategy

The force is equal to the maximum tension, or  $F = 3.0 \times 10^6 \text{ N}$ . The cross-sectional area is  $\pi r^2 = 2.46 \times 10^{-3} \text{ m}^2$ . The equation  $\Delta L = \frac{1}{Y} \frac{F}{A} L_0$  can be used to find the change in length.

### Solution

All quantities are known. Thus,

### Equation:

$$\begin{aligned}\Delta L &= \left( \frac{1}{210 \times 10^9 \text{ N/m}^2} \right) \left( \frac{3.0 \times 10^6 \text{ N}}{2.46 \times 10^{-3} \text{ m}^2} \right) (3020 \text{ m}) \\ &= 18 \text{ m.}\end{aligned}$$

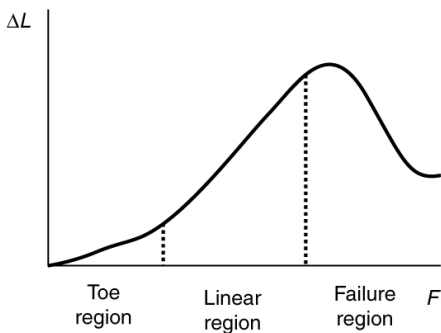
### Discussion

This is quite a stretch, but only about 0.6% of the unsupported length. Effects of temperature upon length might be important in these environments.

Bones, on the whole, do not fracture due to tension or compression. Rather they generally fracture due to sideways impact or bending, resulting in the

bone shearing or snapping. The behavior of bones under tension and compression is important because it determines the load the bones can carry. Bones are classified as weight-bearing structures such as columns in buildings and trees. Weight-bearing structures have special features; columns in building have steel-reinforcing rods while trees and bones are fibrous. The bones in different parts of the body serve different structural functions and are prone to different stresses. Thus the bone in the top of the femur is arranged in thin sheets separated by marrow while in other places the bones can be cylindrical and filled with marrow or just solid. Overweight people have a tendency toward bone damage due to sustained compressions in bone joints and tendons.

Another biological example of Hooke's law occurs in tendons. Functionally, the tendon (the tissue connecting muscle to bone) must stretch easily at first when a force is applied, but offer a much greater restoring force for a greater strain. [\[link\]](#) shows a stress-strain relationship for a human tendon. Some tendons have a high collagen content so there is relatively little strain, or length change; others, like support tendons (as in the leg) can change length up to 10%. Note that this stress-strain curve is nonlinear, since the slope of the line changes in different regions. In the first part of the stretch called the toe region, the fibers in the tendon begin to align in the direction of the stress—this is called *uncrimping*. In the linear region, the fibrils will be stretched, and in the failure region individual fibers begin to break. A simple model of this relationship can be illustrated by springs in parallel: different springs are activated at different lengths of stretch. Examples of this are given in the problems at end of this chapter. Ligaments (tissue connecting bone to bone) behave in a similar way.



Typical stress-strain curve for mammalian tendon. Three regions are shown: (1) toe region (2) linear region, and (3) failure region.

Unlike bones and tendons, which need to be strong as well as elastic, the arteries and lungs need to be very stretchable. The elastic properties of the arteries are essential for blood flow. The pressure in the arteries increases and arterial walls stretch when the blood is pumped out of the heart. When the aortic valve shuts, the pressure in the arteries drops and the arterial walls relax to maintain the blood flow. When you feel your pulse, you are feeling exactly this—the elastic behavior of the arteries as the blood gushes through with each pump of the heart. If the arteries were rigid, you would not feel a pulse. The heart is also an organ with special elastic properties. The lungs expand with muscular effort when we breathe in but relax freely and elastically when we breathe out. Our skins are particularly elastic, especially for the young. A young person can go from 100 kg to 60 kg with no visible sag in their skins. The elasticity of all organs reduces with age. Gradual physiological aging through reduction in elasticity starts in the early 20s.

**Example:**

**Calculating Deformation: How Much Does Your Leg Shorten When You Stand on It?**

Calculate the change in length of the upper leg bone (the femur) when a 70.0 kg man supports 62.0 kg of his mass on it, assuming the bone to be equivalent to a uniform rod that is 40.0 cm long and 2.00 cm in radius.

**Strategy**

The force is equal to the weight supported, or

**Equation:**



$$F = mg = (62.0 \text{ kg})(9.80 \text{ m/s}^2) = 607.6 \text{ N},$$

and the cross-sectional area is  $\pi r^2 = 1.257 \times 10^{-3} \text{ m}^2$ . The equation  $\Delta L = \frac{1}{Y} \frac{F}{A} L_0$  can be used to find the change in length.

### Solution

All quantities except  $\Delta L$  are known. Note that the compression value for Young's modulus for bone must be used here. Thus,

### Equation:

$$\begin{aligned} \Delta L &= \left( \frac{1}{9 \times 10^9 \text{ N/m}^2} \right) \left( \frac{607.6 \text{ N}}{1.257 \times 10^{-3} \text{ m}^2} \right) (0.400 \text{ m}) \\ &= 2 \times 10^{-5} \text{ m}. \end{aligned}$$

### Discussion

This small change in length seems reasonable, consistent with our experience that bones are rigid. In fact, even the rather large forces encountered during strenuous physical activity do not compress or bend bones by large amounts. Although bone is rigid compared with fat or muscle, several of the substances listed in [\[link\]](#) have larger values of Young's modulus  $Y$ . In other words, they are more rigid.

The equation for change in length is traditionally rearranged and written in the following form:

### Equation:

$$\frac{F}{A} = Y \frac{\Delta L}{L_0}.$$

The ratio of force to area,  $\frac{F}{A}$ , is defined as **stress** (measured in  $\text{N/m}^2$ ), and the ratio of the change in length to length,  $\frac{\Delta L}{L_0}$ , is defined as **strain** (a unitless quantity). In other words,

### Equation:

$$\text{stress} = Y \times \text{strain}.$$

In this form, the equation is analogous to Hooke's law, with stress analogous to force and strain analogous to deformation. If we again rearrange this equation to the form

**Equation:**

$$F = YA \frac{\Delta L}{L_0},$$

we see that it is the same as Hooke's law with a proportionality constant

**Equation:**

$$k = \frac{YA}{L_0}.$$

This general idea—that force and the deformation it causes are proportional for small deformations—applies to changes in length, sideways bending, and changes in volume.

**Note:**

Stress

The ratio of force to area,  $\frac{F}{A}$ , is defined as stress measured in N/m<sup>2</sup>.

**Note:**

Strain

The ratio of the change in length to length,  $\frac{\Delta L}{L_0}$ , is defined as strain (a unitless quantity). In other words,

**Equation:**

$$\text{stress} = Y \times \text{strain}.$$

## Sideways Stress: Shear Modulus

[\[link\]](#) illustrates what is meant by a sideways stress or a *shearing force*. Here the deformation is called  $\Delta x$  and it is perpendicular to  $L_0$ , rather than parallel as with tension and compression. Shear deformation behaves similarly to tension and compression and can be described with similar equations. The expression for **shear deformation** is

**Equation:**

$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where  $S$  is the shear modulus (see [\[link\]](#)) and  $F$  is the force applied perpendicular to  $L_0$  and parallel to the cross-sectional area  $A$ . Again, to keep the object from accelerating, there are actually two equal and opposite forces  $F$  applied across opposite faces, as illustrated in [\[link\]](#). The equation is logical—for example, it is easier to bend a long thin pencil (small  $A$ ) than a short thick one, and both are more easily bent than similar steel rods (large  $S$ ).

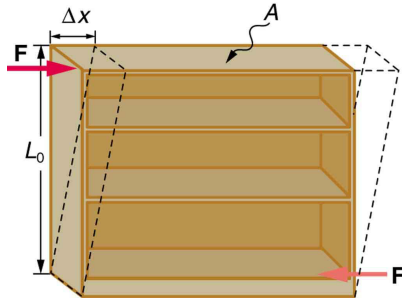
**Note:**

Shear Deformation

**Equation:**

$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where  $S$  is the shear modulus and  $F$  is the force applied perpendicular to  $L_0$  and parallel to the cross-sectional area  $A$ .



Shearing forces are applied perpendicular to the length  $L_0$  and parallel to the area  $A$ , producing a deformation  $\Delta x$ . Vertical forces are not shown, but it should be kept in mind that in addition to the two shearing forces,  $\mathbf{F}$ , there must be supporting forces to keep the object from rotating. The distorting effects of these supporting forces are ignored in this treatment. The weight of the object also is not shown, since it is usually negligible compared with forces large enough to cause significant deformations.

Examination of the shear moduli in [\[link\]](#) reveals some telling patterns. For example, shear moduli are less than Young's moduli for most materials. Bone is a remarkable exception. Its shear modulus is not only greater than its Young's modulus, but it is as large as that of steel. This is why bones are so rigid.

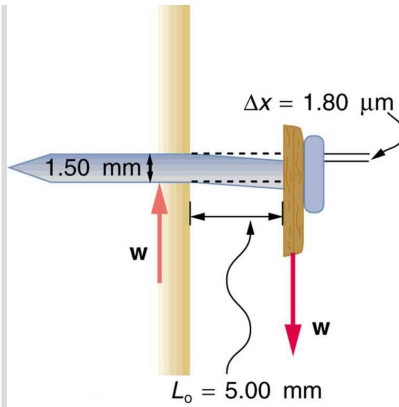
The spinal column (consisting of 26 vertebral segments separated by discs) provides the main support for the head and upper part of the body. The spinal column has normal curvature for stability, but this curvature can be increased, leading to increased shearing forces on the lower vertebrae. Discs are better at withstanding compressional forces than shear forces. Because the spine is not vertical, the weight of the upper body exerts some of both. Pregnant women and people that are overweight (with large abdomens) need to move their shoulders back to maintain balance, thereby increasing the curvature in their spine and so increasing the shear component of the stress. An increased angle due to more curvature increases the shear forces along the plane. These higher shear forces increase the risk of back injury through ruptured discs. The lumbosacral disc (the wedge shaped disc below the last vertebrae) is particularly at risk because of its location.

The shear moduli for concrete and brick are very small; they are too highly variable to be listed. Concrete used in buildings can withstand compression, as in pillars and arches, but is very poor against shear, as might be encountered in heavily loaded floors or during earthquakes. Modern structures were made possible by the use of steel and steel-reinforced concrete. Almost by definition, liquids and gases have shear moduli near zero, because they flow in response to shearing forces.

**Example:**

**Calculating Force Required to Deform: That Nail Does Not Bend Much Under a Load**

Find the mass of the picture hanging from a steel nail as shown in [\[link\]](#), given that the nail bends only  $1.80\text{ }\mu\text{m}$ . (Assume the shear modulus is known to two significant figures.)



Side view of a nail with a picture hung from it. The nail flexes very slightly (shown much larger than actual) because of the shearing effect of the supported weight. Also shown is the upward force of the wall on the nail, illustrating that there are equal and opposite forces applied across opposite cross sections of the nail.

See [\[link\]](#) for a calculation of the mass of the picture.

### Strategy

The force  $F$  on the nail (neglecting the nail's own weight) is the weight of the picture  $w$ . If we can find  $w$ , then the mass of the picture is just  $\frac{w}{g}$ . The equation  $\Delta x = \frac{1}{S} \frac{F}{A} L_0$  can be solved for  $F$ .

**Solution**

Solving the equation  $\Delta x = \frac{1}{S} \frac{F}{A} L_0$  for  $F$ , we see that all other quantities can be found:

**Equation:**

$$F = \frac{SA}{L_0} \Delta x.$$

$S$  is found in [\[link\]](#) and is  $S = 80 \times 10^9 \text{ N/m}^2$ . The radius  $r$  is 0.750 mm (as seen in the figure), so the cross-sectional area is

**Equation:**

$$A = \pi r^2 = 1.77 \times 10^{-6} \text{ m}^2.$$

The value for  $L_0$  is also shown in the figure. Thus,

**Equation:**

$$F = \frac{(80 \times 10^9 \text{ N/m}^2)(1.77 \times 10^{-6} \text{ m}^2)}{(5.00 \times 10^{-3} \text{ m})} (1.80 \times 10^{-6} \text{ m}) = 51 \text{ N}.$$

This 51 N force is the weight  $w$  of the picture, so the picture's mass is

**Equation:**

$$m = \frac{w}{g} = \frac{F}{g} = 5.2 \text{ kg}.$$

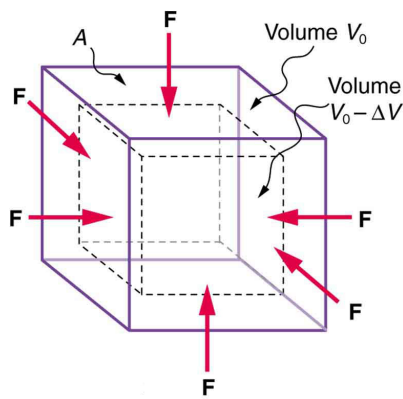
**Discussion**

This is a fairly massive picture, and it is impressive that the nail flexes only  $1.80 \mu\text{m}$ —an amount undetectable to the unaided eye.

## Changes in Volume: Bulk Modulus

An object will be compressed in all directions if inward forces are applied evenly on all its surfaces as in [\[link\]](#). It is relatively easy to compress gases and extremely difficult to compress liquids and solids. For example, air in a

wine bottle is compressed when it is corked. But if you try corking a brim-full bottle, you cannot compress the wine—some must be removed if the cork is to be inserted. The reason for these different compressibilities is that atoms and molecules are separated by large empty spaces in gases but packed close together in liquids and solids. To compress a gas, you must force its atoms and molecules closer together. To compress liquids and solids, you must actually compress their atoms and molecules, and very strong electromagnetic forces in them oppose this compression.



An inward force on all surfaces compresses this cube. Its change in volume is proportional to the force per unit area and its original volume, and is related to the compressibility of the substance.

We can describe the compression or volume deformation of an object with an equation. First, we note that a force “applied evenly” is defined to have



the same stress, or ratio of force to area  $\frac{F}{A}$  on all surfaces. The deformation produced is a change in volume  $\Delta V$ , which is found to behave very similarly to the shear, tension, and compression previously discussed. (This is not surprising, since a compression of the entire object is equivalent to compressing each of its three dimensions.) The relationship of the change in volume to other physical quantities is given by

**Equation:**

$$\Delta V = \frac{1}{B} \frac{F}{A} V_0,$$

where  $B$  is the bulk modulus (see [\[link\]](#)),  $V_0$  is the original volume, and  $\frac{F}{A}$  is the force per unit area applied uniformly inward on all surfaces. Note that no bulk moduli are given for gases.

What are some examples of bulk compression of solids and liquids? One practical example is the manufacture of industrial-grade diamonds by compressing carbon with an extremely large force per unit area. The carbon atoms rearrange their crystalline structure into the more tightly packed pattern of diamonds. In nature, a similar process occurs deep underground, where extremely large forces result from the weight of overlying material. Another natural source of large compressive forces is the pressure created by the weight of water, especially in deep parts of the oceans. Water exerts an inward force on all surfaces of a submerged object, and even on the water itself. At great depths, water is measurably compressed, as the following example illustrates.

**Example:**

**Calculating Change in Volume with Deformation: How Much Is Water Compressed at Great Ocean Depths?**

Calculate the fractional decrease in volume ( $\frac{\Delta V}{V_0}$ ) for seawater at 5.00 km depth, where the force per unit area is  $5.00 \times 10^7 \text{ N/m}^2$ .

**Strategy**

Equation  $\Delta V = \frac{1}{B} \frac{F}{A} V_0$  is the correct physical relationship. All quantities in the equation except  $\frac{\Delta V}{V_0}$  are known.

### **Solution**

Solving for the unknown  $\frac{\Delta V}{V_0}$  gives

**Equation:**

$$\frac{\Delta V}{V_0} = \frac{1}{B} \frac{F}{A}.$$

Substituting known values with the value for the bulk modulus  $B$  from [\[link\]](#),

**Equation:**

$$\begin{aligned} \frac{\Delta V}{V_0} &= \frac{5.00 \times 10^7 \text{ N/m}^2}{2.2 \times 10^9 \text{ N/m}^2} \\ &= 0.023 = 2.3\%. \end{aligned}$$

### **Discussion**

Although measurable, this is not a significant decrease in volume considering that the force per unit area is about 500 atmospheres (1 million pounds per square foot). Liquids and solids are extraordinarily difficult to compress.

Conversely, very large forces are created by liquids and solids when they try to expand but are constrained from doing so—which is equivalent to compressing them to less than their normal volume. This often occurs when a contained material warms up, since most materials expand when their temperature increases. If the materials are tightly constrained, they deform or break their container. Another very common example occurs when water freezes. Water, unlike most materials, expands when it freezes, and it can easily fracture a boulder, rupture a biological cell, or crack an engine block that gets in its way.

Other types of deformations, such as torsion or twisting, behave analogously to the tension, shear, and bulk deformations considered here.

**Note:**

PhET Explorations: Masses & Springs

[https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab\\_en.html](https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab_en.html)

## Section Summary

- Hooke's law is given by

**Equation:**

$$F = k\Delta L,$$

where  $\Delta L$  is the amount of deformation (the change in length),  $F$  is the applied force, and  $k$  is a proportionality constant that depends on the shape and composition of the object and the direction of the force. The relationship between the deformation and the applied force can also be written as

**Equation:**

$$\Delta L = \frac{1}{Y} \frac{F}{A} L_0,$$

where  $Y$  is *Young's modulus*, which depends on the substance,  $A$  is the cross-sectional area, and  $L_0$  is the original length.

- The ratio of force to area,  $\frac{F}{A}$ , is defined as *stress*, measured in  $\text{N/m}^2$ .
- The ratio of the change in length to length,  $\frac{\Delta L}{L_0}$ , is defined as *strain* (a unitless quantity). In other words,

**Equation:**

$$\text{stress} = Y \times \text{strain}.$$

- The expression for shear deformation is

**Equation:**

$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where  $S$  is the shear modulus and  $F$  is the force applied perpendicular to  $L_0$  and parallel to the cross-sectional area  $A$ .

- The relationship of the change in volume to other physical quantities is given by

**Equation:**

$$\Delta V = \frac{1}{B} \frac{F}{A} V_0,$$

where  $B$  is the bulk modulus,  $V_0$  is the original volume, and  $\frac{F}{A}$  is the force per unit area applied uniformly inward on all surfaces.

## Conceptual Questions

**Exercise:**

**Problem:**

The elastic properties of the arteries are essential for blood flow. Explain the importance of this in terms of the characteristics of the flow of blood (pulsating or continuous).

**Exercise:**

**Problem:**

What are you feeling when you feel your pulse? Measure your pulse rate for 10 s and for 1 min. Is there a factor of 6 difference?

**Exercise:**

**Problem:**

Examine different types of shoes, including sports shoes and thongs. In terms of physics, why are the bottom surfaces designed as they are? What differences will dry and wet conditions make for these surfaces?

**Exercise:****Problem:**

Would you expect your height to be different depending upon the time of day? Why or why not?

**Exercise:****Problem:**

Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

**Exercise:****Problem:**

Explain why pregnant women often suffer from back strain late in their pregnancy.

**Exercise:****Problem:**

An old carpenter's trick to keep nails from bending when they are pounded into hard materials is to grip the center of the nail firmly with pliers. Why does this help?

**Exercise:****Problem:**

When a glass bottle full of vinegar warms up, both the vinegar and the glass expand, but vinegar expands significantly more with temperature than glass. The bottle will break if it was filled to its tightly capped lid. Explain why, and also explain how a pocket of air above the vinegar would prevent the break. (This is the function of the air above liquids in glass containers.)

**Problems & Exercises**

**Exercise:****Problem:**

During a circus act, one performer swings upside down hanging from a trapeze holding another, also upside-down, performer by the legs. If the upward force on the lower performer is three times her weight, how much do the bones (the femurs) in her upper legs stretch? You may assume each is equivalent to a uniform rod 35.0 cm long and 1.80 cm in radius. Her mass is 60.0 kg.

---

**Solution:****Equation:**

$$1.90 \times 10^{-3} \text{ cm}$$

**Exercise:****Problem:**

During a wrestling match, a 150 kg wrestler briefly stands on one hand during a maneuver designed to perplex his already moribund adversary. By how much does the upper arm bone shorten in length? The bone can be represented by a uniform rod 38.0 cm in length and 2.10 cm in radius.

**Exercise:****Problem:**

(a) The “lead” in pencils is a graphite composition with a Young’s modulus of about  $1 \times 10^9 \text{ N/m}^2$ . Calculate the change in length of the lead in an automatic pencil if you tap it straight into the pencil with a force of 4.0 N. The lead is 0.50 mm in diameter and 60 mm long. (b) Is the answer reasonable? That is, does it seem to be consistent with what you have observed when using pencils?

---

**Solution:**

(a) 1 mm

(b) This does seem reasonable, since the lead does seem to shrink a little when you push on it.

**Exercise:**

**Problem:**

TV broadcast antennas are the tallest artificial structures on Earth. In 1987, a 72.0-kg physicist placed himself and 400 kg of equipment at the top of one 610-m high antenna to perform gravity experiments. By how much was the antenna compressed, if we consider it to be equivalent to a steel cylinder 0.150 m in radius?

**Exercise:**

**Problem:**

(a) By how much does a 65.0-kg mountain climber stretch her 0.800-cm diameter nylon rope when she hangs 35.0 m below a rock outcropping? (b) Does the answer seem to be consistent with what you have observed for nylon ropes? Would it make sense if the rope were actually a bungee cord?

---

**Solution:**

(a) 9 cm

(b) This seems reasonable for nylon climbing rope, since it is not supposed to stretch that much.

**Exercise:**

**Problem:**

A 20.0-m tall hollow aluminum flagpole is equivalent in stiffness to a solid cylinder 4.00 cm in diameter. A strong wind bends the pole much as a horizontal force of 900 N exerted at the top would. How far to the side does the top of the pole flex?

**Exercise:**

**Problem:**

As an oil well is drilled, each new section of drill pipe supports its own weight and that of the pipe and drill bit beneath it. Calculate the stretch in a new 6.00 m length of steel pipe that supports 3.00 km of pipe having a mass of 20.0 kg/m and a 100-kg drill bit. The pipe is equivalent in stiffness to a solid cylinder 5.00 cm in diameter.

---

**Solution:**

8.59 mm

**Exercise:****Problem:**

Calculate the force a piano tuner applies to stretch a steel piano wire 8.00 mm, if the wire is originally 0.850 mm in diameter and 1.35 m long.

**Exercise:****Problem:**

A vertebra is subjected to a shearing force of 500 N. Find the shear deformation, taking the vertebra to be a cylinder 3.00 cm high and 4.00 cm in diameter.

---

**Solution:****Equation:**

$$1.49 \times 10^{-7} \text{ m}$$

**Exercise:**



**Problem:**

A disk between vertebrae in the spine is subjected to a shearing force of 600 N. Find its shear deformation, taking it to have the shear modulus of  $1 \times 10^9 \text{ N/m}^2$ . The disk is equivalent to a solid cylinder 0.700 cm high and 4.00 cm in diameter.

**Exercise:****Problem:**

When using a pencil eraser, you exert a vertical force of 6.00 N at a distance of 2.00 cm from the hardwood-eraser joint. The pencil is 6.00 mm in diameter and is held at an angle of  $20.0^\circ$  to the horizontal. (a) By how much does the wood flex perpendicular to its length? (b) How much is it compressed lengthwise?

---

**Solution:**

(a)  $3.99 \times 10^{-7} \text{ m}$

(b)  $9.67 \times 10^{-8} \text{ m}$

**Exercise:****Problem:**

To consider the effect of wires hung on poles, we take data from [\[link\]](#), in which tensions in wires supporting a traffic light were calculated. The left wire made an angle  $30.0^\circ$  below the horizontal with the top of its pole and carried a tension of 108 N. The 12.0 m tall hollow aluminum pole is equivalent in stiffness to a 4.50 cm diameter solid cylinder. (a) How far is it bent to the side? (b) By how much is it compressed?

**Exercise:**

**Problem:**

A farmer making grape juice fills a glass bottle to the brim and caps it tightly. The juice expands more than the glass when it warms up, in such a way that the volume increases by 0.2% (that is,  $\Delta V/V_0 = 2 \times 10^{-3}$ ) relative to the space available. Calculate the magnitude of the normal force exerted by the juice per square centimeter if its bulk modulus is  $1.8 \times 10^9 \text{ N/m}^2$ , assuming the bottle does not break. In view of your answer, do you think the bottle will survive?

---

**Solution:**

$4 \times 10^6 \text{ N/m}^2$ . This is about 36 atm, greater than a typical jar can withstand.

**Exercise:****Problem:**

(a) When water freezes, its volume increases by 9.05% (that is,  $\Delta V/V_0 = 9.05 \times 10^{-2}$ ). What force per unit area is water capable of exerting on a container when it freezes? (It is acceptable to use the bulk modulus of water in this problem.) (b) Is it surprising that such forces can fracture engine blocks, boulders, and the like?

**Exercise:****Problem:**

This problem returns to the tightrope walker studied in [\[link\]](#), who created a tension of  $3.94 \times 10^3 \text{ N}$  in a wire making an angle  $5.0^\circ$  below the horizontal with each supporting pole. Calculate how much this tension stretches the steel wire if it was originally 15 m long and 0.50 cm in diameter.

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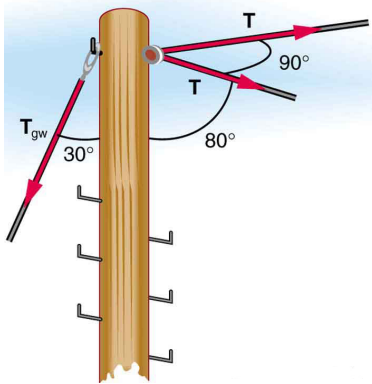
**Solution:**

1.4 cm

## Exercise:

### Problem:

The pole in [\[link\]](#) is at a  $90.0^\circ$  bend in a power line and is therefore subjected to more shear force than poles in straight parts of the line. The tension in each line is  $4.00 \times 10^4 \text{ N}$ , at the angles shown. The pole is 15.0 m tall, has an 18.0 cm diameter, and can be considered to have half the stiffness of hardwood. (a) Calculate the compression of the pole. (b) Find how much it bends and in what direction. (c) Find the tension in a guy wire used to keep the pole straight if it is attached to the top of the pole at an angle of  $30.0^\circ$  with the vertical. (Clearly, the guy wire must be in the opposite direction of the bend.)



This telephone pole is at a  $90^\circ$  bend in a power line. A guy wire is attached to the top of the pole at an angle of  $30^\circ$  with the vertical.

## Glossary

deformation

change in shape due to the application of force

Hooke's law

proportional relationship between the force  $F$  on a material and the deformation  $\Delta L$  it causes,  $F = k\Delta L$

tensile strength

the breaking stress that will cause permanent deformation or fracture of a material

stress

ratio of force to area

strain

ratio of change in length to original length

shear deformation

deformation perpendicular to the original length of an object

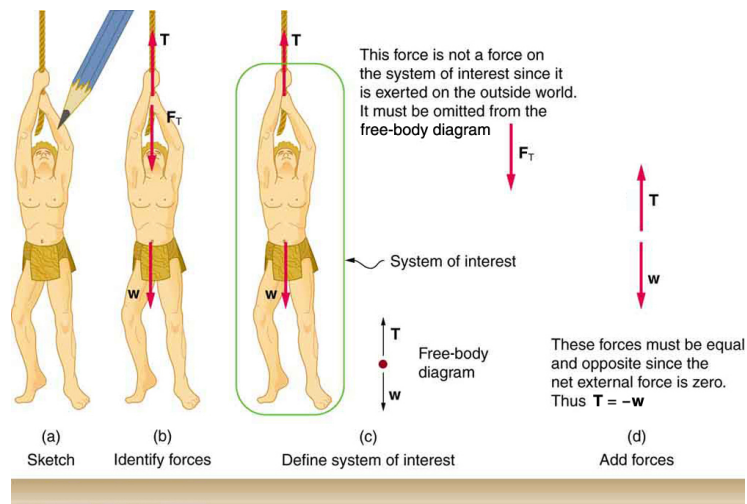
## Problem-Solving Strategies

- Understand and apply a problem-solving procedure to solve problems using Newton's laws of motion.

Success in problem solving is obviously necessary to understand and apply physical principles, not to mention the more immediate need of passing exams. The basics of problem solving, presented earlier in this text, are followed here, but specific strategies useful in applying Newton's laws of motion are emphasized. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, and so the following techniques should reinforce skills you have already begun to develop.

### Problem-Solving Strategy for Newton's Laws of Motion

Step 1. As usual, it is first necessary to identify the physical principles involved. *Once it is determined that Newton's laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation.* Such a sketch is shown in [\[link\]\(a\)](#). Then, as in [\[link\]\(b\)](#), use arrows to represent all forces, label them carefully, and make their lengths and directions correspond to the forces they represent (whenever sufficient information exists).



(a) A sketch of Tarzan hanging from a vine. (b) Arrows are used to represent all forces.  $\mathbf{T}$  is the tension in the vine above Tarzan,  $\mathbf{F}_T$  is the force he exerts on the vine, and  $\mathbf{w}$  is his weight. All other forces, such as the nudge of a breeze, are assumed negligible. (c) Suppose we are given the ape man's mass and asked to find the tension in the vine. We then define the system of interest as shown and draw a free-body diagram.  $\mathbf{F}_T$  is no longer shown, because it is not a force acting on the system of interest; rather,  $\mathbf{F}_T$  acts on the outside world. (d) Showing only the arrows, the head-to-tail method of addition is used. It is apparent that  $\mathbf{T} = -\mathbf{w}$ , if Tarzan is stationary.

Step 2. Identify what needs to be determined and what is known or can be inferred from the problem as stated. That is, make a list of knowns and unknowns. *Then carefully determine the system of interest.* This decision is a crucial step, since Newton's second law involves only external forces. Once the system of interest has been identified, it becomes possible to determine which forces are external and which are internal, a necessary step to

employ Newton's second law. (See [\[link\]\(c\)](#).) Newton's third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated earlier in this chapter, the system of interest depends on what question we need to answer. This choice becomes easier with practice, eventually developing into an almost unconscious process. Skill in clearly defining systems will be beneficial in later chapters as well.

A diagram showing the system of interest and all of the external forces is called a **free-body diagram**. Only forces are shown on free-body diagrams, not acceleration or velocity. We have drawn several of these in worked examples. [\[link\]\(c\)](#) shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Step 3. Once a free-body diagram is drawn, *Newton's second law can be applied to solve the problem*. This is done in [\[link\]\(d\)](#) for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces are parallel—then they add like scalars. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. This is done by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known.

**Note:**

**Applying Newton's Second Law**

Before you write net force equations, it is critical to determine whether the system is accelerating in a particular direction. If the acceleration is zero in a particular direction, then the net force is zero in that direction. Similarly, if the acceleration is nonzero in a particular direction, then the net force is described by the equation:  $F_{\text{net}} = ma$ . For example, if the system is accelerating in the horizontal direction, but it is not accelerating in the vertical direction, then you will have the following conclusions:

**Equation:**

$$F_{\text{net } x} = ma,$$

**Equation:**

$$F_{\text{net } y} = 0.$$

You will need this information in order to determine unknown forces acting in a system.

Step 4. As always, *check the solution to see whether it is reasonable*. In some cases, this is obvious. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving, and with experience it becomes progressively easier to judge whether an answer is reasonable. Another way to check your solution is to check the units. If you are solving for force and end up with units of m/s, then you have made a mistake.

## Section Summary

- To solve problems involving Newton's laws of motion, follow the procedure described:
  1. Draw a sketch of the problem.
  2. Identify known and unknown quantities, and identify the system of interest. Draw a free-body diagram, which is a sketch showing all of the forces acting on an object. The object is represented by a dot, and the forces are represented by vectors extending in different directions from the dot. If vectors act in

directions that are not horizontal or vertical, resolve the vectors into horizontal and vertical components and draw them on the free-body diagram.

3. Write Newton's second law in the horizontal and vertical directions and add the forces acting on the object. If the object does not accelerate in a particular direction (for example, the  $x$ -direction) then  $F_{\text{net } x} = 0$ . If the object does accelerate in that direction,  $F_{\text{net } x} = ma$ .
4. Check your answer. Is the answer reasonable? Are the units correct?

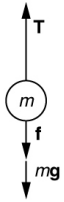
## Problem Exercises

### Exercise:

#### Problem:

A  $5.00 \times 10^5$ -kg rocket is accelerating straight up. Its engines produce  $1.250 \times 10^7$  N of thrust, and air resistance is  $4.50 \times 10^6$  N. What is the rocket's acceleration? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

#### Solution:



Using the free-body diagram:

$$F_{\text{net}} = T - f - mg = ma,$$

so that

$$a = \frac{T - f - mg}{m} = \frac{1.250 \times 10^7 \text{ N} - 4.50 \times 10^6 \text{ N} - (5.00 \times 10^5 \text{ kg})(9.80 \text{ m/s}^2)}{5.00 \times 10^5 \text{ kg}} = 6.20 \text{ m/s}^2.$$

### Exercise:

#### Problem:

The wheels of a midsize car exert a force of 2100 N backward on the road to accelerate the car in the forward direction. If the force of friction including air resistance is 250 N and the acceleration of the car is  $1.80 \text{ m/s}^2$ , what is the mass of the car plus its occupants? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. For this situation, draw a free-body diagram and write the net force equation.

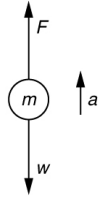
### Exercise:

#### Problem:

Calculate the force a 70.0-kg high jumper must exert on the ground to produce an upward acceleration 4.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

#### Solution:

Use Newton's laws of motion.



Given :  $a = 4.00g = (4.00)(9.80 \text{ m/s}^2) = 39.2 \text{ m/s}^2$ ;  $m = 70.0 \text{ kg}$ ,

Find:  $F$ .

$$\sum F = +F - w = ma, \text{ so } F = ma + w = ma + mg = m(a + g).$$

that

$$F = (70.0 \text{ kg})[(39.2 \text{ m/s}^2) + (9.80 \text{ m/s}^2)] = 3.43 \times 10^3 \text{ N}.$$

The force exerted by the high-jumper is actually down on the ground, but  $F$  is up from the ground and makes him jump.

This result is reasonable, since it is quite possible for a person to exert a force of the magnitude of  $10^3 \text{ N}$ .

### Exercise:

#### Problem:

When landing after a spectacular somersault, a 40.0-kg gymnast decelerates by pushing straight down on the mat. Calculate the force she must exert if her deceleration is 7.00 times the acceleration due to gravity.

Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

### Exercise:

#### Problem:

A freight train consists of two  $8.00 \times 10^4$ -kg engines and 45 cars with average masses of  $5.50 \times 10^4 \text{ kg}$ . (a) What force must each engine exert backward on the track to accelerate the train at a rate of  $5.00 \times 10^{-2} \text{ m/s}^2$  if the force of friction is  $7.50 \times 10^5 \text{ N}$ , assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

#### Solution:

(a)  $4.41 \times 10^5 \text{ N}$

(b)  $1.50 \times 10^5 \text{ N}$

### Exercise:

#### Problem:

Commercial airplanes are sometimes pushed out of the passenger loading area by a tractor. (a) An 1800-kg tractor exerts a force of  $1.75 \times 10^4 \text{ N}$  backward on the pavement, and the system experiences forces resisting motion that total 2400 N. If the acceleration is  $0.150 \text{ m/s}^2$ , what is the mass of the airplane? (b) Calculate the force exerted by the tractor on the airplane, assuming 2200 N of the friction is experienced by the airplane. (c) Draw two sketches showing the systems of interest used to solve each part, including the free-body diagrams for each.

### Exercise:



**Problem:**

A 1100-kg car pulls a boat on a trailer. (a) What total force resists the motion of the car, boat, and trailer, if the car exerts a 1900-N force on the road and produces an acceleration of  $0.550 \text{ m/s}^2$ ? The mass of the boat plus trailer is 700 kg. (b) What is the force in the hitch between the car and the trailer if 80% of the resisting forces are experienced by the boat and trailer?

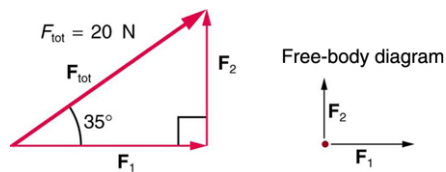
**Solution:**

(a) 910 N

(b)  $1.11 \times 10^3 \text{ N}$

**Exercise:****Problem:**

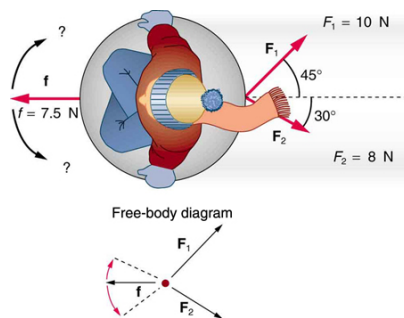
(a) Find the magnitudes of the forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  that add to give the total force  $\mathbf{F}_{\text{tot}}$  shown in [\[link\]](#). This may be done either graphically or by using trigonometry. (b) Show graphically that the same total force is obtained independent of the order of addition of  $\mathbf{F}_1$  and  $\mathbf{F}_2$ . (c) Find the direction and magnitude of some other pair of vectors that add to give  $\mathbf{F}_{\text{tot}}$ . Draw these to scale on the same drawing used in part (b) or a similar picture.

**Exercise:****Problem:**

Two children pull a third child on a snow saucer sled exerting forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  as shown from above in [\[link\]](#). Find the acceleration of the 49.00-kg sled and child system. Note that the direction of the frictional force is unspecified; it will be in the opposite direction of the sum of  $\mathbf{F}_1$  and  $\mathbf{F}_2$ .

**Solution:**

$a = 0.139 \text{ m/s}$ ,  $\theta = 12.4^\circ$  north of east



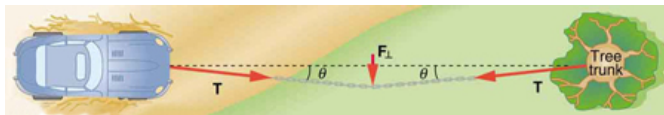
An overhead view of the horizontal forces acting on a

child's snow saucer sled.

### Exercise:

#### Problem:

Suppose your car was mired deeply in the mud and you wanted to use the method illustrated in [\[link\]](#) to pull it out. (a) What force would you have to exert perpendicular to the center of the rope to produce a force of 12,000 N on the car if the angle is  $2.00^\circ$ ? In this part, explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. (b) Real ropes stretch under such forces. What force would be exerted on the car if the angle increases to  $7.00^\circ$  and you still apply the force found in part (a) to its center?



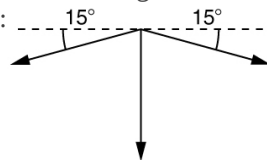
### Exercise:

#### Problem:

What force is exerted on the tooth in [\[link\]](#) if the tension in the wire is 25.0 N? Note that the force applied to the tooth is smaller than the tension in the wire, but this is necessitated by practical considerations of how force can be applied in the mouth. Explicitly show how you follow steps in the Problem-Solving Strategy for Newton's laws of motion.

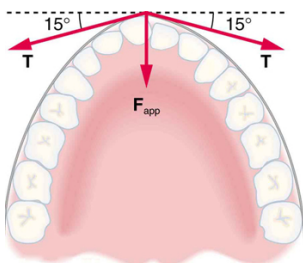
#### Solution:

Use Newton's laws since we are looking for forces.  
Draw a free-body diagram:



The tension is given as  $T = 25.0 \text{ N}$ . Find  $F_{\text{app}}$ . Using Newton's laws gives:  $\Sigma F_y = 0$ , so that y-components of the two tensions:  $F_{\text{app}} = 2 T \sin \theta = 2(25.0 \text{ N})\sin(15^\circ) =$

This seems reasonable, since the applied tensions should be greater than the force applied to the tooth.



Braces are used to apply forces to teeth to realign them. Shown in this figure are the tensions applied by the wire to the protruding tooth. The total force applied to the tooth by the wire,  $\mathbf{F}_{\text{app}}$ , points straight toward the back of the mouth.

**Exercise:**

**Problem:**

[\[link\]](#) shows Superhero and Trusty Sidekick hanging motionless from a rope. Superhero's mass is 90.0 kg, while Trusty Sidekick's is 55.0 kg, and the mass of the rope is negligible. (a) Draw a free-body diagram of the situation showing all forces acting on Superhero, Trusty Sidekick, and the rope. (b) Find the tension in the rope above Superhero. (c) Find the tension in the rope between Superhero and Trusty Sidekick. Indicate on your free-body diagram the system of interest used to solve each part.



Superhero and Trusty Sidekick hang motionless on a rope as they try to figure out what to do next. Will the tension be the same everywhere in the rope?

**Exercise:****Problem:**

A nurse pushes a cart by exerting a force on the handle at a downward angle  $35.0^\circ$  below the horizontal. The loaded cart has a mass of 28.0 kg, and the force of friction is 60.0 N. (a) Draw a free-body diagram for the system of interest. (b) What force must the nurse exert to move at a constant velocity?

**Exercise:****Problem:**

**Construct Your Own Problem** Consider the tension in an elevator cable during the time the elevator starts from rest and accelerates its load upward to some cruising velocity. Taking the elevator and its load to be the system of interest, draw a free-body diagram. Then calculate the tension in the cable. Among the things to consider are the mass of the elevator and its load, the final velocity, and the time taken to reach that velocity.

**Exercise:****Problem:**

**Construct Your Own Problem** Consider two people pushing a toboggan with four children on it up a snow-covered slope. Construct a problem in which you calculate the acceleration of the toboggan and its load. Include a free-body diagram of the appropriate system of interest as the basis for your analysis. Show vector forces and their components and explain the choice of coordinates. Among the things to be considered are the forces exerted by those pushing, the angle of the slope, and the masses of the toboggan and children.

**Exercise:****Problem:**

**Unreasonable Results** (a) Repeat [\[link\]](#), but assume an acceleration of  $1.20 \text{ m/s}^2$  is produced. (b) What is unreasonable about the result? (c) Which premise is unreasonable, and why is it unreasonable?

**Exercise:****Problem:**

**Unreasonable Results** (a) What is the initial acceleration of a rocket that has a mass of  $1.50 \times 10^6 \text{ kg}$  at takeoff, the engines of which produce a thrust of  $2.00 \times 10^6 \text{ N}$ ? Do not neglect gravity. (b) What is unreasonable about the result? (This result has been unintentionally achieved by several real rockets.) (c) Which premise is unreasonable, or which premises are inconsistent? (You may find it useful to compare this problem to the rocket problem earlier in this section.)

## Further Applications of Newton's Laws of Motion

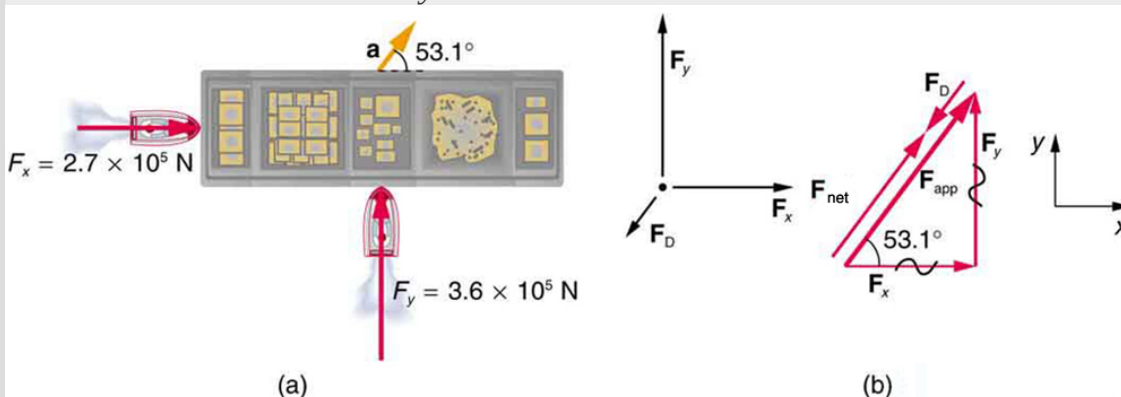
- Apply problem-solving techniques to solve for quantities in more complex systems of forces.
- Integrate concepts from kinematics to solve problems using Newton's laws of motion.

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

### Example:

#### Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in [\[link\]](#). The first tugboat exerts a force of  $2.7 \times 10^5$  N in the x-direction, and the second tugboat exerts a force of  $3.6 \times 10^5$  N in the y-direction.



(a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the  $x$ - and  $y$ -axes are in the same direction as  $\mathbf{F}_x$  and  $\mathbf{F}_y$ . The problem quickly becomes a one-dimensional problem along the direction of  $\mathbf{F}_{app}$ , since friction is in the direction opposite to  $\mathbf{F}_{app}$ .

If the mass of the barge is  $5.0 \times 10^6$  kg and its acceleration is observed to be  $7.5 \times 10^{-2} \text{ m/s}^2$  in the direction shown, what is the drag force of the water on the

barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

**Strategy**

The directions and magnitudes of acceleration and the applied forces are given in [\[link\]\(a\)](#). We will define the total force of the tugboats on the barge as  $\mathbf{F}_{\text{app}}$  so that:

**Equation:**

$$\mathbf{F}_{\text{app}} = \mathbf{F}_x + \mathbf{F}_y$$

Since the barge is flat bottomed, the drag of the water  $\mathbf{F}_D$  will be in the direction opposite to  $\mathbf{F}_{\text{app}}$ , as shown in the free-body diagram in [\[link\]\(b\)](#). The system of interest here is the barge, since the forces on *it* are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force  $\mathbf{F}_{\text{app}}$ , and then apply Newton's second law to solve for the drag force  $\mathbf{F}_D$ .

**Solution**

Since  $\mathbf{F}_x$  and  $\mathbf{F}_y$  are perpendicular, the magnitude and direction of  $\mathbf{F}_{\text{app}}$  are easily found. First, the resultant magnitude is given by the Pythagorean theorem:

**Equation:**

$$\begin{aligned} F_{\text{app}} &= \sqrt{F_x^2 + F_y^2} \\ F_{\text{app}} &= \sqrt{(2.7 \times 10^5 \text{ N})^2 + (3.6 \times 10^5 \text{ N})^2} = 4.5 \times 10^5 \text{ N}. \end{aligned}$$

The angle is given by

**Equation:**

$$\begin{aligned} \theta &= \tan^{-1} \left( \frac{F_y}{F_x} \right) \\ \theta &= \tan^{-1} \left( \frac{3.6 \times 10^5 \text{ N}}{2.7 \times 10^5 \text{ N}} \right) = 53^\circ, \end{aligned}$$

which we know, because of Newton's first law, is the same direction as the acceleration.  $\mathbf{F}_D$  is in the opposite direction of  $\mathbf{F}_{\text{app}}$ , since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as  $\mathbf{F}_{\text{app}}$ , but its magnitude is slightly less than  $\mathbf{F}_{\text{app}}$ . The problem is now one-dimensional. From [\[link\]\(b\)](#), we can see that

**Equation:**

$$F_{\text{net}} = F_{\text{app}} - F_D.$$

But Newton's second law states that

**Equation:**

$$F_{\text{net}} = ma.$$

Thus,

**Equation:**

$$F_{\text{app}} - F_{\text{D}} = ma.$$

This can be solved for the magnitude of the drag force of the water  $F_{\text{D}}$  in terms of known quantities:

**Equation:**

$$F_{\text{D}} = F_{\text{app}} - ma.$$

Substituting known values gives

**Equation:**

$$F_{\text{D}} = (4.5 \times 10^5 \text{ N}) - (5.0 \times 10^6 \text{ kg})(7.5 \times 10^{-2} \text{ m/s}^2) = 7.5 \times 10^4 \text{ N}.$$

The direction of  $\mathbf{F}_{\text{D}}$  has already been determined to be in the direction opposite to  $\mathbf{F}_{\text{app}}$ , or at an angle of  $53^\circ$  south of west.

**Discussion**

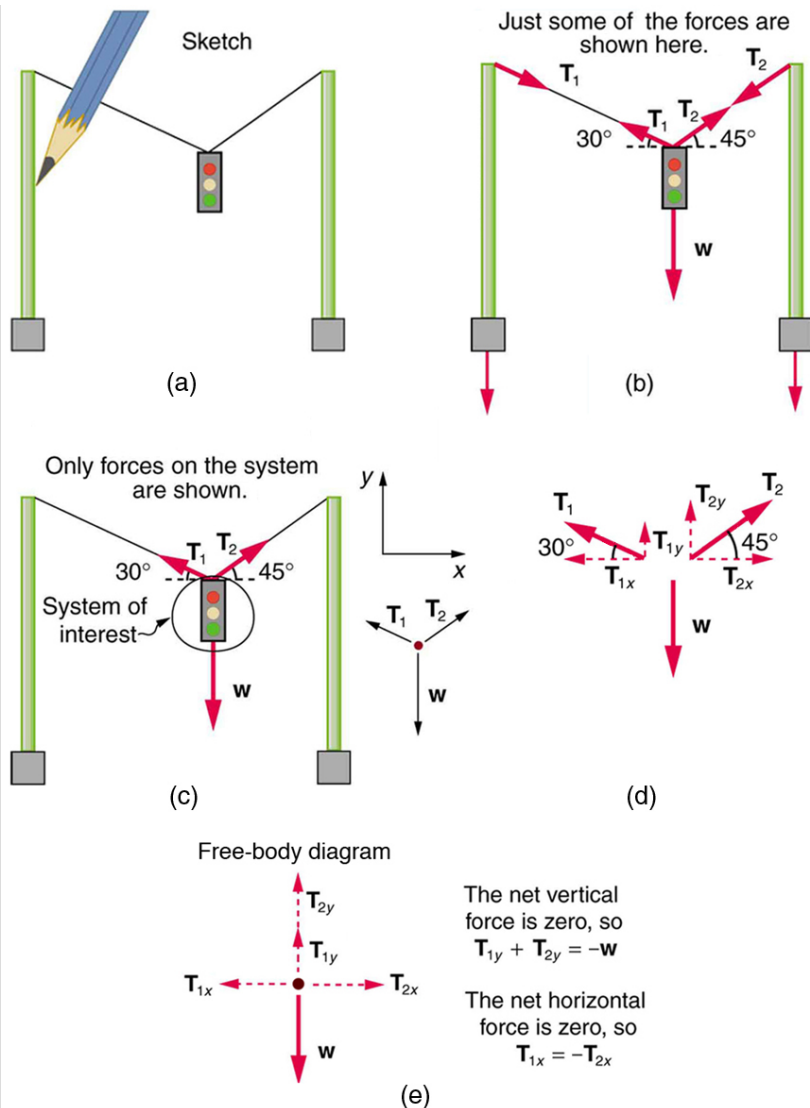
The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where  $F_{\text{D}}$  is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

**Example:**

**Different Tensions at Different Angles**

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in [\[link\]](#). Find the tension in each wire, neglecting the masses of the wires.



A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical ( $y$ ) and horizontal ( $x$ ) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

## Strategy



The system of interest is the traffic light, and its free-body diagram is shown in [\[link\]](#) (c). The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem ( $T_1$  and  $T_2$ ), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

**Solution**

First consider the horizontal or  $x$ -axis:

**Equation:**

$$F_{\text{net}x} = T_{2x} - T_{1x} = 0.$$

Thus, as you might expect,

**Equation:**

$$T_{1x} = T_{2x}.$$

This gives us the following relationship between  $T_1$  and  $T_2$ :

**Equation:**

$$T_1 \cos(30^\circ) = T_2 \cos(45^\circ).$$

Thus,

**Equation:**

$$T_2 = (1.225)T_1.$$

Note that  $T_1$  and  $T_2$  are not equal in this case, because the angles on either side are not equal. It is reasonable that  $T_2$  ends up being greater than  $T_1$ , because it is exerted more vertically than  $T_1$ .

Now consider the force components along the vertical or  $y$ -axis:

**Equation:**

$$F_{\text{net}y} = T_{1y} + T_{2y} - w = 0.$$

This implies

**Equation:**

$$T_{1y} + T_{2y} = w.$$

Substituting the expressions for the vertical components gives

**Equation:**

$$T_1 \sin (30^\circ) + T_2 \sin (45^\circ) = w.$$

There are two unknowns in this equation, but substituting the expression for  $T_2$  in terms of  $T_1$  reduces this to one equation with one unknown:

**Equation:**

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg,$$

which yields

**Equation:**

$$(1.366)T_1 = (15.0 \text{ kg})(9.80 \text{ m/s}^2).$$

Solving this last equation gives the magnitude of  $T_1$  to be

**Equation:**

$$T_1 = 108 \text{ N}.$$

Finally, the magnitude of  $T_2$  is determined using the relationship between them,  $T_2 = 1.225 T_1$ , found above. Thus we obtain

**Equation:**

$$T_2 = 132 \text{ N}.$$

### Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

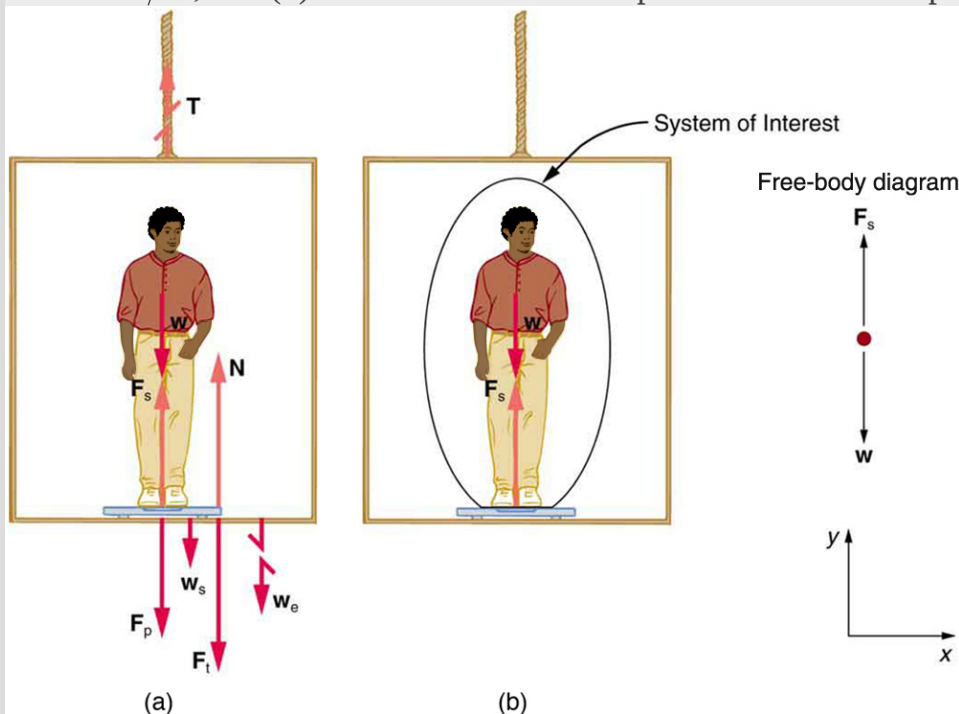
The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

### Example:

#### What Does the Bathroom Scale Read in an Elevator?

[\[link\]](#) shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate

of  $1.20 \text{ m/s}^2$ , and (b) if the elevator moves upward at a constant speed of  $1 \text{ m/s}$ .



(a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale.  $\mathbf{T}$  is the tension in the supporting cable,  $\mathbf{w}$  is the weight of the person,  $\mathbf{w}_s$  is the weight of the scale,  $\mathbf{w}_e$  is the weight of the elevator,  $\mathbf{F}_s$  is the force of the scale on the person,  $\mathbf{F}_p$  is the force of the person on the scale,  $\mathbf{F}_t$  is the force of the scale on the floor of the elevator, and  $\mathbf{N}$  is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

### Strategy

If the scale is accurate, its reading will equal  $F_p$ , the magnitude of the force the person exerts downward on it. [\[link\]](#)(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in [\[link\]](#)(b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight  $\mathbf{w}$  and the upward force of the scale  $\mathbf{F}_s$ . According to Newton's third law  $\mathbf{F}_p$  and  $\mathbf{F}_s$  are

equal in magnitude and opposite in direction, so that we need to find  $F_s$  in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,  
**Equation:**

$$F_{\text{net}} = ma.$$

From the free-body diagram we see that  $F_{\text{net}} = F_s - w$ , so that  
**Equation:**

$$F_s - w = ma.$$

Solving for  $F_s$  gives an equation with only one unknown:  
**Equation:**

$$F_s = ma + w,$$

or, because  $w = mg$ , simply  
**Equation:**

$$F_s = ma + mg.$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

#### **Solution for (a)**

In this part of the problem,  $a = 1.20 \text{ m/s}^2$ , so that

**Equation:**

$$F_s = (75.0 \text{ kg})(1.20 \text{ m/s}^2) + (75.0 \text{ kg})(9.80 \text{ m/s}^2),$$

yielding

**Equation:**

$$F_s = 825 \text{ N}.$$

#### **Discussion for (a)**

This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

**Equation:**

$$\begin{aligned} F_{\text{net}} &= ma = 0 = F_s - w \\ F_s &= w = mg \\ F_s &= (75.0 \text{ kg})(9.80 \text{ m/s}^2) \\ F_s &= 735 \text{ N}. \end{aligned}$$

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

**Solution for (b)**

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because  $a = \frac{\Delta v}{\Delta t}$ , and  $\Delta v = 0$ .

Thus,

**Equation:**

$$F_s = ma + mg = 0 + mg.$$

Now

**Equation:**

$$F_s = (75.0 \text{ kg})(9.80 \text{ m/s}^2),$$

which gives

**Equation:**

$$F_s = 735 \text{ N}.$$

**Discussion for (b)**

The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward,  $a$  is negative, and the scale reading is *less* than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at  $g$ , then the scale reading will be zero and the person will *appear* to be weightless.

## **Integrating Concepts: Newton's Laws of Motion and Kinematics**

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton's laws of motion can also be integrated with other concepts that have been discussed previously in this text to

solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

**Problem-Solving Strategy**

- Step 1. *Identify which physical principles are involved.* Listing the givens and the quantities to be calculated will allow you to identify the principles involved.
- Step 2. *Solve the problem using strategies outlined in the text.* If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

**Example:**

**What Force Must a Soccer Player Exert to Reach Top Speed?**

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player’s mass is 70.0 kg, and air resistance is negligible.

**Strategy**

To solve an integrated concept problem, first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers acceleration along a straight line. This is a topic of kinematics. Part (b) deals with force, a dynamics topic found in this chapter.

The following solutions to each part of the example illustrate how the specific

problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

**Solution for (a)**

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is  $\Delta v = 8.00$  m/s. We are given the elapsed time, and so  $\Delta t = 2.50$  s. The unknown is acceleration, which can be found from its definition:

**Equation:**

$$a = \frac{\Delta v}{\Delta t}.$$

Substituting the known values yields

**Equation:**

$$\begin{aligned} a &= \frac{8.00 \text{ m/s}}{2.50 \text{ s}} \\ &= 3.20 \text{ m/s}^2. \end{aligned}$$

**Discussion for (a)**

This is an attainable acceleration for an athlete in good condition.

**Solution for (b)**

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player's acceleration and are given his mass, we can use Newton's second law to find the force exerted. That is,

**Equation:**

$$F_{\text{net}} = ma.$$

Substituting the known values of  $m$  and  $a$  gives

**Equation:**

$$\begin{aligned} F_{\text{net}} &= (70.0 \text{ kg})(3.20 \text{ m/s}^2) \\ &= 224 \text{ N}. \end{aligned}$$

**Discussion for (b)**

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these

techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

## Summary

- Newton's laws of motion can be applied in numerous situations to solve problems of motion.
- Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether  $F_{\text{net}} = ma$  or  $F_{\text{net}} = 0$ .
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics in order to solve these problems of motion.

## Conceptual Questions

### Exercise:

#### Problem:

To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at  $g$ . Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

### Exercise:

#### Problem:

A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.



## Problem Exercises

### Exercise:

#### Problem:

A flea jumps by exerting a force of  $1.20 \times 10^{-5}$  N straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of  $0.500 \times 10^{-6}$  N on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is  $6.00 \times 10^{-7}$  kg. Do not neglect the gravitational force.

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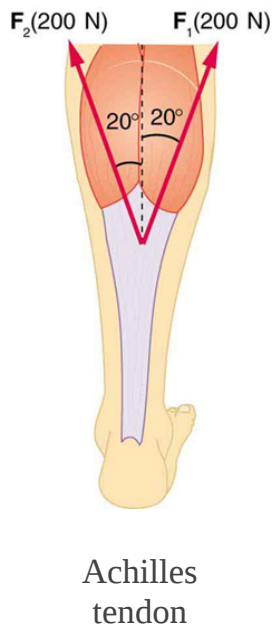
#### Solution:

$10.2 \text{ m/s}^2$ ,  $4.67^\circ$  from vertical

### Exercise:

#### Problem:

Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in [\[link\]](#). (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?

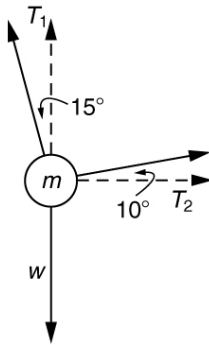


### Exercise:

**Problem:**

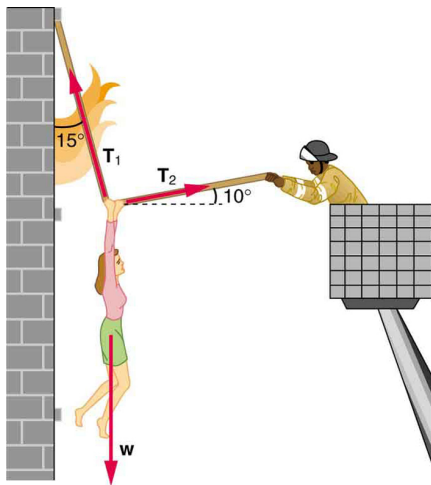
A 76.0-kg person is being pulled away from a burning building as shown in [\[link\]](#). Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

---

**Solution:**

$$T_1 = 736 \text{ N}$$

$$T_2 = 194 \text{ N}$$



The force  $T_2$  needed to hold steady the person being rescued from the fire is less than her weight and less than the force  $T_1$  in the other rope, since the more

vertical rope supports a greater part of her weight (a vertical force).

**Exercise:**

**Problem:**

**Integrated Concepts** A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow him if he was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)

**Exercise:**

**Problem:**

**Integrated Concepts** When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel?

---

**Solution:**

(a) 7.43 m/s

(b) 2.97 m

**Exercise:**

**Problem:**

**Integrated Concepts** A large rocket has a mass of  $2.00 \times 10^6$  kg at takeoff, and its engines produce a thrust of  $3.50 \times 10^7$  N. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust? (c) In reality, the mass of a rocket decreases significantly as its fuel is consumed. Describe qualitatively how this affects the acceleration and time for this motion.

**Exercise:**

**Problem:**

**Integrated Concepts** A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in part (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110 kg.

---

**Solution:**

(a) 4.20 m/s

(b)  $29.4 \text{ m/s}^2$

(c)  $4.31 \times 10^3 \text{ N}$

**Exercise:****Problem:**

**Integrated Concepts** A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell's velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.

**Exercise:****Problem:**

**Integrated Concepts** Repeat [\[link\]](#) for a shell fired at an angle  $10.0^\circ$  from the vertical.

---

**Solution:**

(a) 47.1 m/s

(b)  $2.47 \times 10^3 \text{ m/s}^2$

(c)  $6.18 \times 10^3 \text{ N}$  . The average force is 252 times the shell's weight.

**Exercise:**

**Problem:**

**Integrated Concepts** An elevator filled with passengers has a mass of 1700 kg. (a) The elevator accelerates upward from rest at a rate of  $1.20 \text{ m/s}^2$  for 1.50 s. Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at constant velocity for 8.50 s. What is the tension in the cable during this time? (c) The elevator decelerates at a rate of  $0.600 \text{ m/s}^2$  for 3.00 s. What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

**Exercise:**

**Problem:**

**Unreasonable Results** (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of  $0.400 \text{ m/s}^2$  for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

**Exercise:**

**Problem:**

**Unreasonable Results** A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

## Introduction to Uniform Circular Motion and Gravitation

class="introduction"

This  
Australian  
Grand Prix  
Formula 1  
race car  
moves in a  
circular  
path as it  
makes the  
turn. Its  
wheels also  
spin rapidly  
—the latter  
completing  
many  
revolutions,  
the former  
only part of  
one (a  
circular  
arc). The  
same  
physical  
principles  
are  
involved in  
each.  
(credit:  
Richard  
Munckton)



Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. In some ways, this chapter is a continuation of [Dynamics: Newton's Laws of Motion](#) as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, motion in a circular path at constant speed. Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics we group under the name *rotation*. Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving along ice.

## **Glossary**

uniform circular motion

the motion of an object in a circular path at constant speed

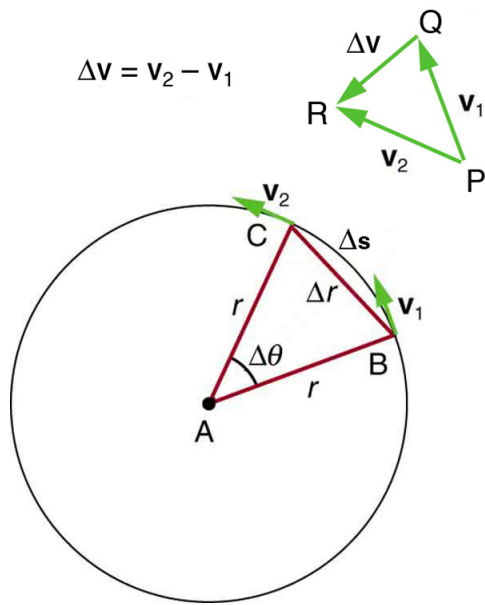


## Centripetal Acceleration

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

[\[link\]](#) shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration**( $a_c$ ); centripetal means “toward the center” or “center seeking.”



The directions of the velocity of an object at two different points are shown, and the change in velocity  $\Delta\mathbf{v}$  is seen to point directly toward the center of curvature. (See small inset.) Because  $\mathbf{a}_c = \Delta\mathbf{v}/\Delta t$ , the acceleration is also toward the center;  $\mathbf{a}_c$  is called centripetal acceleration. (Because  $\Delta\theta$  is very small, the arc length  $\Delta s$  is equal to the chord length  $\Delta r$  for small time differences.)

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii  $r$  and  $\Delta s$  are similar. Both the

triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds  $v_1 = v_2 = v$ . Using the properties of two similar triangles, we obtain

**Equation:**

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}.$$

Acceleration is  $\Delta v / \Delta t$ , and so we first solve this expression for  $\Delta v$ :

**Equation:**

$$\Delta v = \frac{v}{r} \Delta s.$$

Then we divide this by  $\Delta t$ , yielding

**Equation:**

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}.$$

Finally, noting that  $\Delta v / \Delta t = a_c$  and that  $\Delta s / \Delta t = v$ , the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

**Equation:**

$$a_c = \frac{v^2}{r},$$

which is the acceleration of an object in a circle of radius  $r$  at a speed  $v$ . So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that  $a_c$  is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that  $a_c$  is greater for tighter turns, as you have probably noticed.

It is also useful to express  $a_c$  in terms of angular velocity. Substituting  $v = r\omega$  into the above expression, we find  $a_c = (r\omega)^2/r = r\omega^2$ . We can express the magnitude of centripetal acceleration using either of two equations:

**Equation:**

$$a_c = \frac{v^2}{r}; \quad a_c = r\omega^2.$$

Recall that the direction of  $a_c$  is toward the center. You may use whichever expression is more convenient, as illustrated in examples below.

A **centrifuge** (see [\[link\]](#)b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity ( $g$ ); maximum centripetal acceleration of several hundred thousand  $g$  is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth's gravity.

**Example:**

**How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?**

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See [\[link\]](#)(a).

**Strategy**

Because  $v$  and  $r$  are given, the first expression in  $a_c = \frac{v^2}{r}$ ;  $a_c = r\omega^2$  is the most convenient to use.

**Solution**

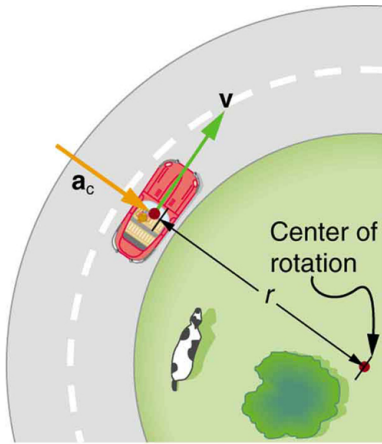
Entering the given values of  $v = 25.0 \text{ m/s}$  and  $r = 500 \text{ m}$  into the first expression for  $a_c$  gives

**Equation:**

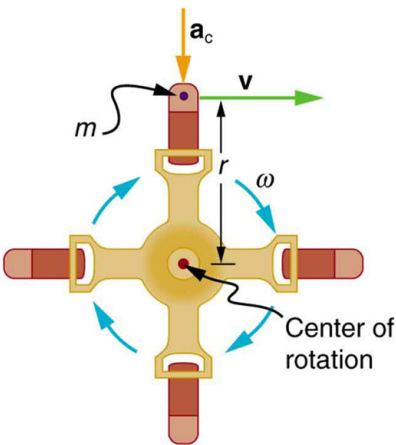
$$a_c = \frac{v^2}{r} = \frac{(25.0 \text{ m/s})^2}{500 \text{ m}} = 1.25 \text{ m/s}^2.$$

**Discussion**

To compare this with the acceleration due to gravity ( $g = 9.80 \text{ m/s}^2$ ), we take the ratio of  $a_c/g = (1.25 \text{ m/s}^2) / (9.80 \text{ m/s}^2) = 0.128$ . Thus,  $a_c = 0.128 g$  and is noticeable especially if you were not wearing a seat belt.



(a) Car around corner



(b) Centrifuge

(a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in [\[link\]](#). (b) A particle of mass in a centrifuge is rotating at constant angular velocity . It

must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in [\[link\]](#).

**Example:**

**How Big Is the Centripetal Acceleration in an Ultracentrifuge?**

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an **ultracentrifuge** spinning at  $7.5 \times 10^4$  rev/min. Determine the ratio of this acceleration to that due to gravity. See [\[link\]](#)(b).

**Strategy**

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity  $\omega$ . Because  $r$  is given, we can use the second expression in the equation  $a_c = \frac{v^2}{r}$ ;  $a_c = r\omega^2$  to calculate the centripetal acceleration.

**Solution**

To convert  $7.50 \times 10^4$  rev/min to radians per second, we use the facts that one revolution is  $2\pi$ rad and one minute is 60.0 s. Thus,

**Equation:**

$$\omega = 7.50 \times 10^4 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60.0 \text{ s}} = 7854 \text{ rad/s}.$$

Now the centripetal acceleration is given by the second expression in

$$a_c = \frac{v^2}{r}; a_c = r\omega^2 \text{ as}$$

**Equation:**

$$a_c = r\omega^2.$$

Converting 7.50 cm to meters and substituting known values gives

**Equation:**

$$a_c = (0.0750 \text{ m})(7854 \text{ rad/s})^2 = 4.63 \times 10^6 \text{ m/s}^2.$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of  $a_c$  to  $g$  yields

**Equation:**

$$\frac{a_c}{g} = \frac{4.63 \times 10^6}{9.80} = 4.72 \times 10^5.$$

### Discussion

This last result means that the centripetal acceleration is 472,000 times as strong as  $g$ . It is no wonder that such high  $\omega$  centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In [Centripetal Force](#), we will consider the forces involved in circular motion.

### Note:

PhET Explorations: Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

<https://archive.cnx.org/specials/317a2b1e-2fbd-11e5-99b5-e38ffb545fe6/ladybug-motion/#sim-ladybug-motion>



## Section Summary

- Centripetal acceleration  $a_c$  is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity  $v$  and has the magnitude  
**Equation:**

$$a_c = \frac{v^2}{r}; a_c = r\omega^2.$$

- The unit of centripetal acceleration is  $\text{m/s}^2$ .

## Conceptual Questions

### Exercise:

#### Problem:

Can centripetal acceleration change the speed of circular motion? Explain.

## Problem Exercises

### Exercise:

#### Problem:

A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

---

#### Solution:

12.9 rev/min

### Exercise:

**Problem:**

A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a radius of curvature of 30 m. If he completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of his centripetal acceleration as he runs the curved portion of the track?

**Exercise:****Problem:**

Taking the age of Earth to be about  $4 \times 10^9$  years and assuming its orbital radius of  $1.5 \times 10^{11}$  m has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun).

---

**Solution:**

$$4 \times 10^{21} \text{ m}$$

**Exercise:****Problem:**

The propeller of a World War II fighter plane is 2.30 m in diameter.

(a) What is its angular velocity in radians per second if it spins at 1200 rev/min?

(b) What is the linear speed of its tip at this angular velocity if the plane is stationary on the tarmac?

(c) What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of  $g$ .

**Exercise:**

**Problem:**

An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

- (a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of  $g$ .
  - (b) What is the linear speed of a point on its edge?
- 

**Solution:**

- a)  $3.47 \times 10^4 \text{ m/s}^2$ ,  $3.55 \times 10^3 g$
- b) 51.1 m/s

**Exercise:****Problem:**

Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

- (a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.
- (b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

**Exercise:**

**Problem:** Olympic ice skaters are able to spin at about 5 rev/s.

- (a) What is their angular velocity in radians per second?
- (b) What is the centripetal acceleration of the skater's nose if it is 0.120 m from the axis of rotation?

(c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?

(d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

---

**Solution:**

a) 31.4 rad/s

b) 118 m/s

c) 384 m/s

d) The centripetal acceleration felt by Olympic skaters is 12 times larger than the acceleration due to gravity. That's quite a lot of acceleration in itself. The centripetal acceleration felt by Button's nose was 39.2 times larger than the acceleration due to gravity. It is no wonder that he ruptured small blood vessels in his spins.

**Exercise:**

**Problem:**

What percentage of the acceleration at Earth's surface is the acceleration due to gravity at the position of a satellite located 300 km above Earth?

**Exercise:**

**Problem:**

Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:

(a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50,000 rev/min.

(b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth's orbit and approximate it as being circular).

---

**Solution:**

a) 0.524 km/s

b) 29.7 km/s

**Exercise:**

**Problem:**

A rotating space station is said to create “artificial gravity”—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an “artificial gravity” of  $9.80 \text{ m/s}^2$  at the rim?

**Exercise:**

**Problem:**

At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.

(a) At how many rev/min are the tires rotating?

(b) What is the centripetal acceleration at the edge of the tire?

(c) With what force must a determined  $1.00 \times 10^{-15} \text{ kg}$  bacterium cling to the rim?

(d) Take the ratio of this force to the bacterium's weight.

---

**Solution:**

- (a)  $1.35 \times 10^3$  rpm
- (b)  $8.47 \times 10^3$  m/s<sup>2</sup>
- (c)  $8.47 \times 10^{-12}$  N
- (d) 865

**Exercise:**

**Problem: Integrated Concepts**

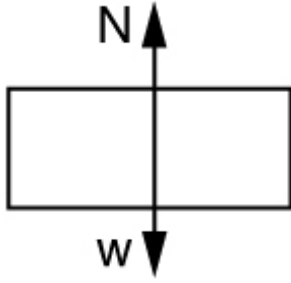
Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity.

- (a) Assuming negligible friction, find the speed of the riders at the bottom of its arc, given the system's center of mass travels in an arc having a radius of 14.0 m and the riders are near the center of mass.
- (b) What is the centripetal acceleration at the bottom of the arc?
- (c) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.
- (d) Find the force exerted by the ride on a 60.0 kg rider and compare it to her weight.
- (e) Discuss whether the answer seems reasonable.

---

**Solution:**

- (a) 16.6 m/s
- (b)  $19.6$  m/s<sup>2</sup>
- (c)



(d)  $1.76 \times 10^3 \text{ N}$  or  $3.00 w$ , that is, the normal force (upward) is three times her weight.

(e) This answer seems reasonable, since she feels like she's being forced into the chair MUCH stronger than just by gravity.

**Exercise:**

**Problem: Unreasonable Results**

A mother pushes her child on a swing so that his speed is  $9.00 \text{ m/s}$  at the lowest point of his path. The swing is suspended  $2.00 \text{ m}$  above the child's center of mass.

(a) What is the magnitude of the centripetal acceleration of the child at the low point?

(b) What is the magnitude of the force the child exerts on the seat if his mass is  $18.0 \text{ kg}$ ?

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent?

---

**Solution:**

a)  $40.5 \text{ m/s}^2$

b)  $905 \text{ N}$

c) The force in part (b) is very large. The acceleration in part (a) is too much, about 4 g.

d) The speed of the swing is too large. At the given velocity at the bottom of the swing, there is enough kinetic energy to send the child all the way over the top, ignoring friction.

## **Glossary**

centripetal acceleration

the acceleration of an object moving in a circle, directed toward the center

ultracentrifuge

a centrifuge optimized for spinning a rotor at very high speeds



## Centripetal Force

- Calculate coefficient of friction on a car tire.
- Calculate ideal speed and angle of a car on a turn.

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's second law of motion, net force is mass times acceleration: net  $F = ma$ . For uniform circular motion, the acceleration is the centripetal acceleration —  $a = a_c$ . Thus, the magnitude of centripetal force  $F_c$  is

**Equation:**

$$F_c = ma_c.$$

By using the expressions for centripetal acceleration  $a_c$  from  $a_c = \frac{v^2}{r}$ ;  $a_c = r\omega^2$ , we get two expressions for the centripetal force  $F_c$  in terms of mass, velocity, angular velocity, and radius of curvature:

**Equation:**

$$F_c = m\frac{v^2}{r}; F_c = mr\omega^2.$$

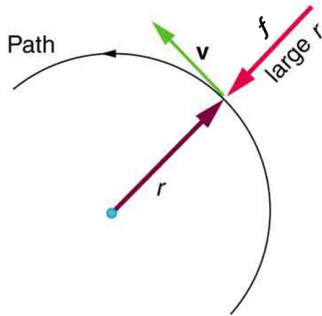
You may use whichever expression for centripetal force is more convenient. Centripetal force  $F_c$  is always perpendicular to the path and pointing to the center of curvature, because  $\mathbf{a}_c$  is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for  $r$ , you get

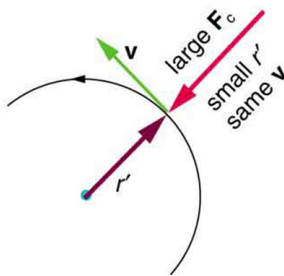
**Equation:**

$$r = \frac{mv^2}{F_c}.$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve.



$f = F_c$  is parallel to  $a_c$  since  $F_c = ma_c$



The frictional force  
supplies the  
centripetal force  
and is numerically  
equal to it.

Centripetal force is  
perpendicular to  
velocity and causes  
uniform circular  
motion. The larger  
the  $F_c$ , the smaller  
the radius of  
curvature  $r$  and the  
sharper the curve.  
The second curve

has the same  $v$ , but  
a larger  $F_c$   
produces a smaller  
 $r$ !

**Example:**

**What Coefficient of Friction Do Car Tires Need on a Flat Curve?**

(a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve at 25.0 m/s.

(b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (see [\[link\]](#)).

**Strategy and Solution for (a)**

We know that  $F_c = \frac{mv^2}{r}$ . Thus,

**Equation:**

$$F_c = \frac{mv^2}{r} = \frac{(900 \text{ kg})(25.0 \text{ m/s})^2}{(500 \text{ m})} = 1125 \text{ N}.$$

**Strategy for (b)**

[\[link\]](#) shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is  $\mu_s N$ , where  $\mu_s$  is the static coefficient of friction and  $N$  is the normal force. The normal force equals the car's weight on level ground, so that  $N = mg$ . Thus the centripetal force in this situation is

**Equation:**

$$F_c = f = \mu_s N = \mu_s mg.$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for  $F_c$  from the equation

**Equation:**

$$\left. \begin{aligned} F_c &= m \frac{v^2}{r} \\ F_c &= mr\omega^2 \end{aligned} \right\},$$

**Equation:**

$$m \frac{v^2}{r} = \mu_s mg.$$

We solve this for  $\mu_s$ , noting that mass cancels, and obtain

**Equation:**

$$\mu_s = \frac{v^2}{rg}.$$

**Solution for (b)**

Substituting the knowns,

**Equation:**

$$\mu_s = \frac{(25.0 \text{ m/s})^2}{(500 \text{ m})(9.80 \text{ m/s}^2)} = 0.13.$$

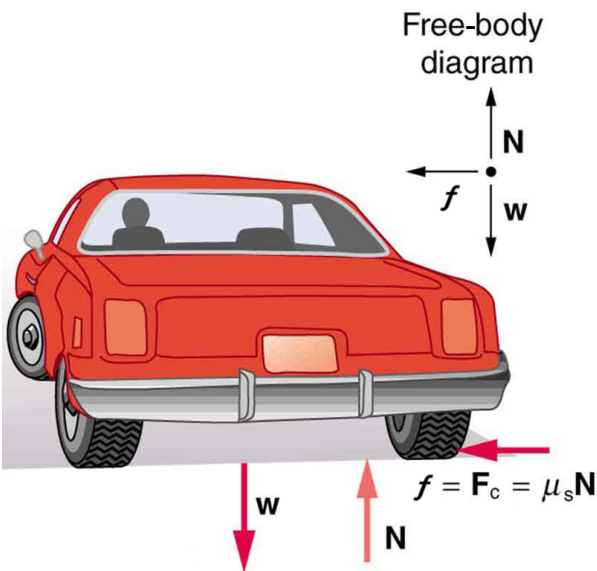
(Because coefficients of friction are approximate, the answer is given to only two digits.)

**Discussion**

We could also solve part (a) using the first expression in  $\left. \begin{aligned} F_c &= m \frac{v^2}{r} \\ F_c &= mr\omega^2 \end{aligned} \right\},$

because  $m, v$ , and  $r$  are given. The coefficient of friction found in part (b) is much smaller than is typically found between tires and roads. The car will still negotiate the curve if the coefficient is greater than 0.13, because static friction is a responsive force, being able to assume a value less than but no more than  $\mu_s N$ . A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is

less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less as will be discussed below.



This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road.

A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See [\[link\]](#). The greater the angle  $\theta$ , the faster you can

take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an “ideally banked curve,” the angle  $\theta$  is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for  $\theta$  for an ideally banked curve and consider an example related to it.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force  $N$  in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

[\[link\]](#) shows a free body diagram for a car on a frictionless banked curve. If the angle  $\theta$  is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight  $\mathbf{w}$  and the normal force of the road  $\mathbf{N}$ . (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude  $mv^2/r$ .

Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is,

**Equation:**

$$N \sin \theta = \frac{mv^2}{r}.$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is  $N \cos \theta$ , and the only other vertical force is the car’s weight. These must be equal in magnitude; thus,

**Equation:**

$$N \cos \theta = mg.$$

Now we can combine the last two equations to eliminate  $N$  and get an expression for  $\theta$ , as desired. Solving the second equation for  $N = mg/(\cos \theta)$ , and substituting this into the first yields

**Equation:**

$$mg \frac{\sin \theta}{\cos \theta} = \frac{mv^2}{r}$$

**Equation:**

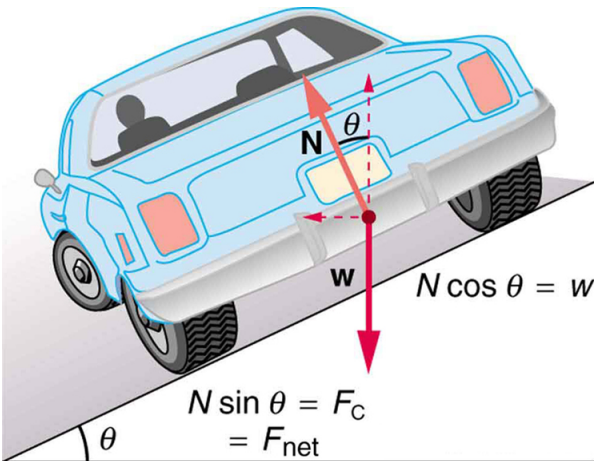
$$\begin{aligned} mg \tan(\theta) &= \frac{mv^2}{r} \\ \tan \theta &= \frac{v^2}{rg} \end{aligned}$$

Taking the inverse tangent gives

**Equation:**

$$\theta = \tan^{-1} \left( \frac{v^2}{rg} \right) \text{ (ideally banked curve, no friction).}$$

This expression can be understood by considering how  $\theta$  depends on  $v$  and  $r$ . A large  $\theta$  will be obtained for a large  $v$  and a small  $r$ . That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that  $\theta$  does not depend on the mass of the vehicle.



The car on this banked curve is moving away and turning to the left.

### Example:

#### What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at  $65.0^\circ$  should be driven if the road is frictionless.

#### Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

#### Solution

Starting with

#### Equation:

$$\tan \theta = \frac{v^2}{rg}$$



we get

**Equation:**

$$v = (rg \tan \theta)^{1/2}.$$

Noting that  $\tan 65.0^\circ = 2.14$ , we obtain

**Equation:**

$$\begin{aligned} v &= \left[ (100 \text{ m})(9.80 \text{ m/s}^2)(2.14) \right]^{1/2} \\ &= 45.8 \text{ m/s.} \end{aligned}$$

### **Discussion**

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Calculations similar to those in the preceding examples can be performed for a host of interesting situations in which centripetal force is involved—a number of these are presented in this chapter's Problems and Exercises.

### **Note:**

#### **Take-Home Experiment**

Ask a friend or relative to swing a golf club or a tennis racquet. Take appropriate measurements to estimate the centripetal acceleration of the end of the club or racquet. You may choose to do this in slow motion.

### **Note:**

#### **PhET Explorations: Gravity and Orbits**

Move the sun, earth, moon and space station to see how it affects their gravitational forces and orbital paths. Visualize the sizes and distances between different heavenly bodies, and turn off gravity to see what would happen without it!

## Section Summary

- Centripetal force  $F_c$  is any force causing uniform circular motion. It is a “center-seeking” force that always points toward the center of rotation. It is perpendicular to linear velocity  $v$  and has magnitude

**Equation:**

$$F_c = ma_c,$$

which can also be expressed as

**Equation:**

$$\left. \begin{array}{l} F_c = m \frac{v^2}{r} \\ \text{or} \\ F_c = mr\omega^2 \end{array} \right\}$$

## Conceptual Questions

**Exercise:**

**Problem:**

If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

**Exercise:**

**Problem:**

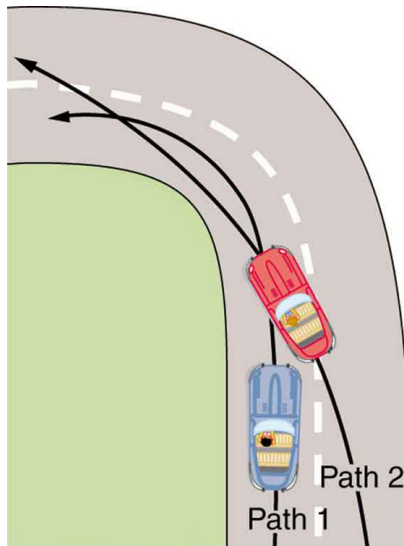
Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

**Exercise:****Problem:**

If centripetal force is directed toward the center, why do you feel that you are ‘thrown’ away from the center as a car goes around a curve? Explain.

**Exercise:****Problem:**

Race car drivers routinely cut corners as shown in [\[link\]](#). Explain how this allows the curve to be taken at the greatest speed.



Two paths around a race track curve are shown. Race car drivers will take the inside path (called cutting the corner)

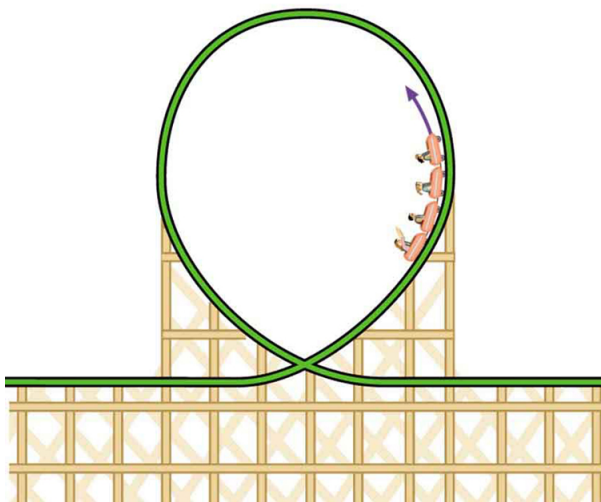
whenever possible  
because it allows  
them to take the  
curve at the highest  
speed.

**Exercise:**

**Problem:**

A number of amusement parks have rides that make vertical loops like the one shown in [\[link\]](#). For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

- (a) The car goes over the top at faster than this speed?
- (b) The car goes over the top at slower than this speed?



Amusement rides with a vertical  
loop are an example of a form  
of curved motion.

**Exercise:****Problem:**

What is the direction of the force exerted by the car on the passenger as the car goes over the top of the amusement ride pictured in [\[link\]](#) under the following circumstances:

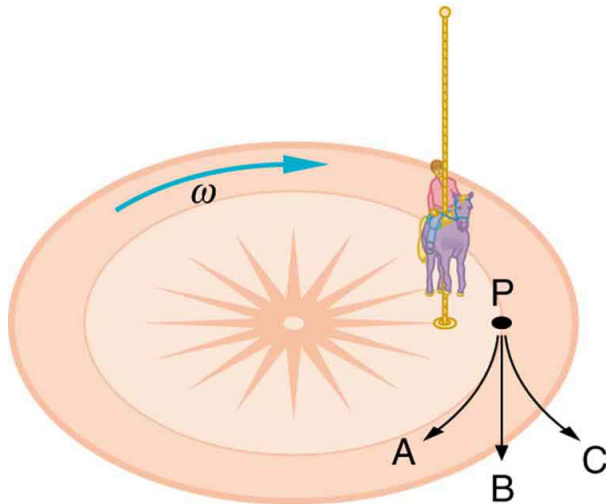
- (a) The car goes over the top at such a speed that the gravitational force is the only force acting?
- (b) The car goes over the top faster than this speed?
- (c) The car goes over the top slower than this speed?

**Exercise:****Problem:**

As a skater forms a circle, what force is responsible for making her turn? Use a free body diagram in your answer.

**Exercise:****Problem:**

Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown in [\[link\]](#) will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



Merry-go-round's rotating  
frame of reference

A child riding on a merry-go-round releases her lunch box at point P. This is a view from above the clockwise rotation.

Assuming it slides with negligible friction, will it follow path A, B, or C, as viewed from Earth's frame of reference?

What will be the shape of the path it leaves in the dust on the merry-go-round?

### Exercise:

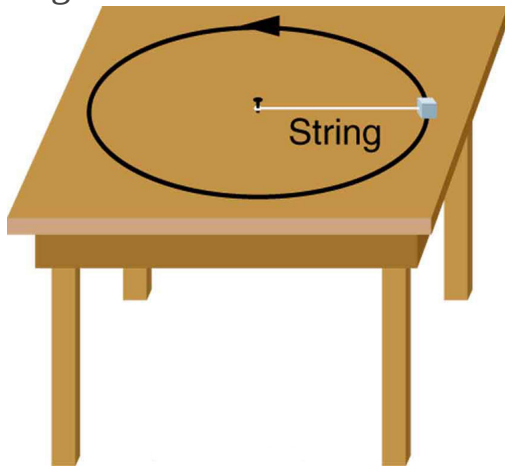
#### Problem:

Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

### Exercise:

**Problem:**

Suppose a mass is moving in a circular path on a frictionless table as shown in figure. In the Earth's frame of reference, there is no centrifugal force pulling the mass away from the centre of rotation, yet there is a very real force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.



A mass attached to a nail on a frictionless table moves in a circular path. The force stretching the string is real and not fictional. What is the physical origin of the force on the string?

**Problems Exercise****Exercise:**

**Problem:**

(a) A 22.0 kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force must she exert to stay on if she is 1.25 m from its center?

(b) What centripetal force does she need to stay on an amusement park merry-go-round that rotates at 3.00 rev/min if she is 8.00 m from its center?

(c) Compare each force with her weight.

---

**Solution:**

a) 483 N

b) 17.4 N

c) 2.24 times her weight, 0.0807 times her weight

**Exercise:****Problem:**

Calculate the centripetal force on the end of a 100 m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.

**Exercise:****Problem:**

What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?

---

**Solution:**

4.14°

**Exercise:**



**Problem:**

What is the ideal speed to take a 100 m radius curve banked at a  $20.0^\circ$  angle?

**Exercise:****Problem:**

- (a) What is the radius of a bobsled turn banked at  $75.0^\circ$  and taken at 30.0 m/s, assuming it is ideally banked?
  - (b) Calculate the centripetal acceleration.
  - (c) Does this acceleration seem large to you?
- 

**Solution:**

- a) 24.6 m
- b)  $36.6 \text{ m/s}^2$
- c)  $a_c = 3.73 g$ . This does not seem too large, but it is clear that bobsledders feel a lot of force on them going through sharply banked turns.

**Exercise:****Problem:**

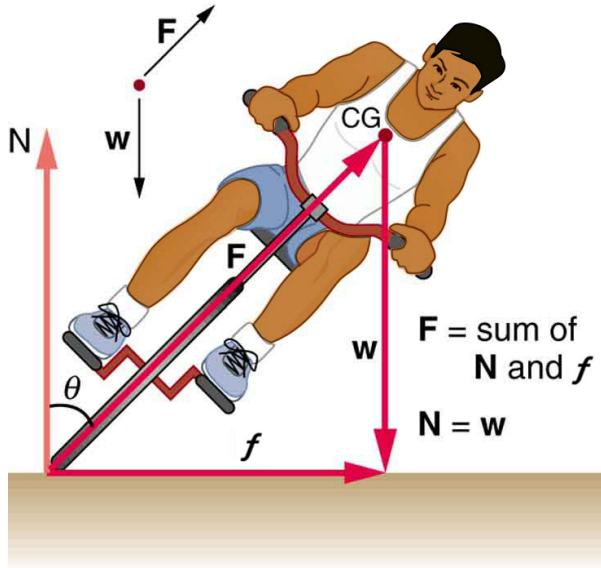
Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen in [\[link\]](#). To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force), and the vertical normal force (which must equal the system's weight).

- (a) Show that  $\theta$  (as defined in the figure) is related to the speed  $v$  and radius of curvature  $r$  of the turn in the same way as for an ideally

banked roadway—that is,  $\theta = \tan^{-1} v^2 / rg$

(b) Calculate  $\theta$  for a 12.0 m/s turn of radius 30.0 m (as in a race).

Free-body diagram



A bicyclist negotiating a turn on level ground must lean at the correct angle—the ability to do this becomes instinctive. The force of the ground on the wheel needs to be on a line through the center of gravity. The net external force on the system is the centripetal force. The vertical component of the force on the wheel cancels the weight of the system while its horizontal component must supply the centripetal force.

This process produces a relationship among the angle  $\theta$ , the speed  $v$ , and the radius of curvature  $r$  of the turn similar to

that for the ideal banking of  
roadways.

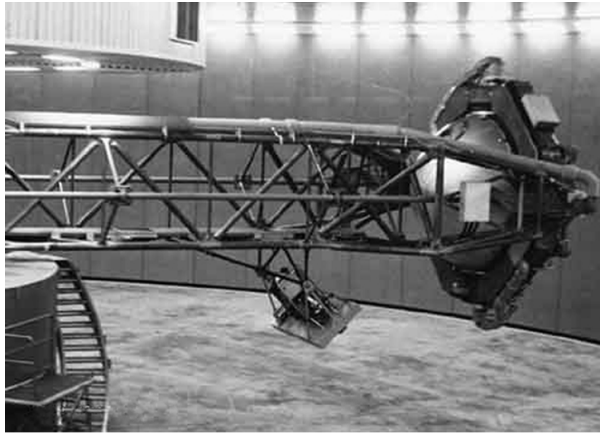
**Exercise:**

**Problem:**

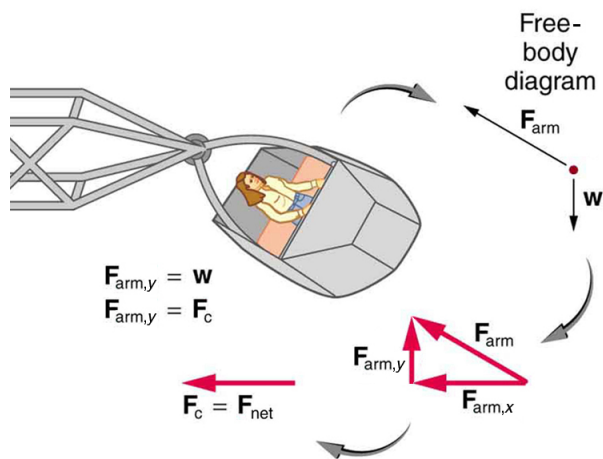
A large centrifuge, like the one shown in [\[link\]](#)(a), is used to expose aspiring astronauts to accelerations similar to those experienced in rocket launches and atmospheric reentries.

(a) At what angular velocity is the centripetal acceleration  $10\ g$  if the rider is 15.0 m from the center of rotation?

(b) The rider's cage hangs on a pivot at the end of the arm, allowing it to swing outward during rotation as shown in [\[link\]](#)(b). At what angle  $\theta$  below the horizontal will the cage hang when the centripetal acceleration is  $10\ g$  ? (Hint: The arm supplies centripetal force and supports the weight of the cage. Draw a free body diagram of the forces to see what the angle  $\theta$  should be.)



(a) NASA centrifuge and ride



(b)

(a) NASA centrifuge used to subject trainees to accelerations similar to those experienced in rocket launches and reentries.

(credit: NASA) (b) Rider in cage showing how the cage pivots outward during rotation.

This allows the total force exerted on the rider by the cage to be along its axis at all times.

---

**Solution:**

a) 2.56 rad/s

b)  $5.71^\circ$

**Exercise:**

**Problem: Integrated Concepts**

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a real problem on icy mountain roads). (a) Calculate the ideal speed to take a 100 m radius curve banked at  $15.0^\circ$ . (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?

---

**Solution:**

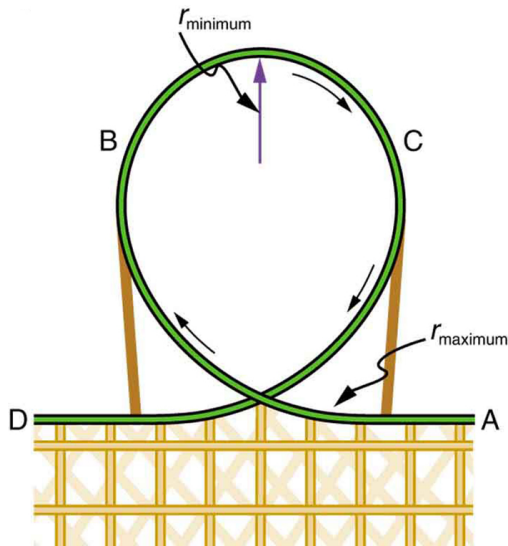
a) 16.2 m/s

b) 0.234

**Exercise:**

**Problem:**

Modern roller coasters have vertical loops like the one shown in [\[link\]](#). The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?



Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradually decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maximum at the top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a disadvantage discovered long ago in railroad curve design. With a small radius of curvature at the top, the centripetal acceleration can more easily be kept greater than  $g$  so that the passengers do not lose contact with their seats nor do they

need seat belts to keep  
them in place.

### **Exercise:**

#### **Problem: Unreasonable Results**

- (a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s.
  - (b) What is unreasonable about the result?
  - (c) Which premises are unreasonable or inconsistent?
- 

#### **Solution:**

- a) 1.84
- b) A coefficient of friction this much greater than 1 is unreasonable .
- c) The assumed speed is too great for the tight curve.

### **Glossary**

centripetal force

any net force causing uniform circular motion

ideal banking

the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

ideal speed

the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road

ideal angle

the angle at which a car can turn safely on a steep curve, which is in proportion to the ideal speed

banked curve

the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve



## Newton's Universal Law of Gravitation

- Explain Earth's gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Examine the Cavendish experiment

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth's gravity on us. An apple falls from a tree because of the same force acting a few meters above Earth's surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

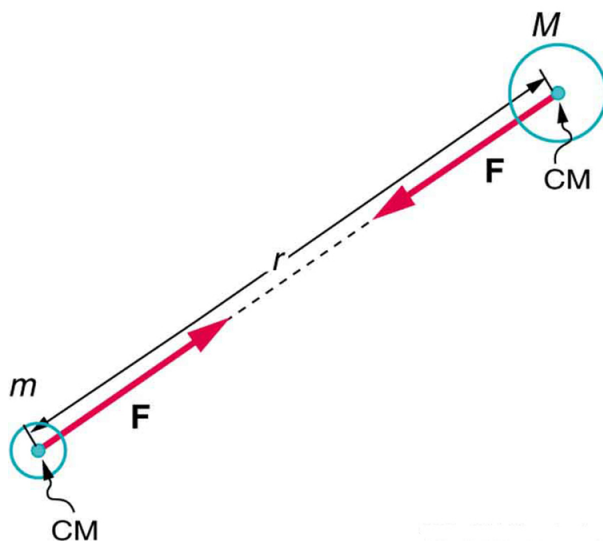
Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See [\[link\]](#). But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton's contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others.



According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton's apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton's

universal law of  
gravitation and his  
laws of motion  
answered very old  
questions about nature  
and gave tremendous  
support to the notion  
of underlying  
simplicity and unity in  
nature. Scientists still  
expect underlying  
simplicity to emerge  
from their ongoing  
inquiries into nature.

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton's third law.

**Note:**

**Misconception Alert**

The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the **center of mass** (CM), which will be further explored in [Linear Momentum and Collisions](#). For two bodies having masses  $m$  and  $M$  with a distance  $r$  between their centers of mass, the equation for Newton's universal law of gravitation is

**Equation:**

$$F = G \frac{mM}{r^2},$$

where  $F$  is the magnitude of the gravitational force and  $G$  is a proportionality factor called the **gravitational constant**.  $G$  is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

**Equation:**

$$G = 6.674 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$$

in SI units. Note that the units of  $G$  are such that a force in newtons is obtained from  $F = G \frac{mM}{r^2}$ , when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of  $6.674 \times 10^{-11}$  N. This is an extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the *entire Earth* on us with a mass of  $6 \times 10^{24}$  kg.

Recall that the acceleration due to gravity  $g$  is about  $9.80 \text{ m/s}^2$  on Earth. We can now determine why this is so. The weight of an object  $mg$  is the gravitational force between it and Earth. Substituting  $mg$  for  $F$  in Newton's universal law of gravitation gives

**Equation:**

$$mg = G \frac{mM}{r^2},$$

where  $m$  is the mass of the object,  $M$  is the mass of Earth, and  $r$  is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See [\[link\]](#). The mass  $m$  of the object cancels, leaving an equation for  $g$ :

**Equation:**

$$g = G \frac{M}{r^2}.$$

Substituting known values for Earth's mass and radius (to three significant figures),

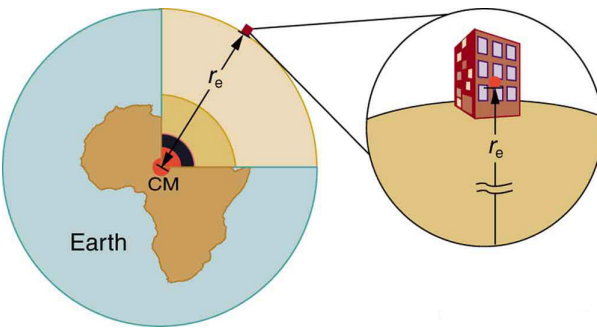
**Equation:**

$$g = \left( 6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{5.98 \times 10^{24} \text{ kg}}{(6.38 \times 10^6 \text{ m})^2},$$

and we obtain a value for the acceleration of a falling body:

**Equation:**

$$g = 9.80 \text{ m/s}^2.$$



The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.

This is the expected value *and is independent of the body's mass*. Newton's law of gravitation takes Galileo's observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

**Note:**

**Take-Home Experiment**

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.

**Note:****Making Connections**

Attempts are still being made to understand the gravitational force. As we shall see in [Particle Physics](#), modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed “pretty nearly.”

**Example:****Earth’s Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path**

- (a) Find the acceleration due to Earth’s gravity at the distance of the Moon.
- (b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth’s gravity that you have just found.

**Strategy for (a)**

This calculation is the same as the one finding the acceleration due to gravity at Earth’s surface, except that  $r$  is the distance from the center of Earth to the center of the Moon. The radius of the Moon’s nearly circular orbit is  $3.84 \times 10^8$  m.

**Solution for (a)**

Substituting known values into the expression for  $g$  found above, remembering that  $M$  is the mass of Earth not the Moon, yields

**Equation:**

$$\begin{aligned}
 g &= G \frac{M}{r^2} = \left( 6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{5.98 \times 10^{24} \text{ kg}}{(3.84 \times 10^8 \text{ m})^2} \\
 &= 2.70 \times 10^{-3} \text{ m/s}^2
 \end{aligned}$$

### Strategy for (b)

Centripetal acceleration can be calculated using either form of **Equation:**

$$\left. \begin{aligned} a_c &= \frac{v^2}{r} \\ a_c &= r\omega^2 \end{aligned} \right\}$$

We choose to use the second form:

**Equation:**

$$a_c = r\omega^2,$$

where  $\omega$  is the angular velocity of the Moon about Earth.

### Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon's orbit is 27.3 days, (d) and using

**Equation:**

$$1 \text{ d} \times 24 \frac{\text{hr}}{\text{d}} \times 60 \frac{\text{min}}{\text{hr}} \times 60 \frac{\text{s}}{\text{min}} = 86,400 \text{ s}$$

we see that

**Equation:**

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ rad}}{(27.3 \text{ d})(86,400 \text{ s/d})} = 2.66 \times 10^{-6} \frac{\text{rad}}{\text{s}}.$$

The centripetal acceleration is

**Equation:**

$$\begin{aligned}
 a_c &= r\omega^2 = (3.84 \times 10^8 \text{ m})(2.66 \times 10^{-6} \text{ rad/s})^2 \\
 &= 2.72 \times 10^{-3} \text{ m/s}^2
 \end{aligned}$$

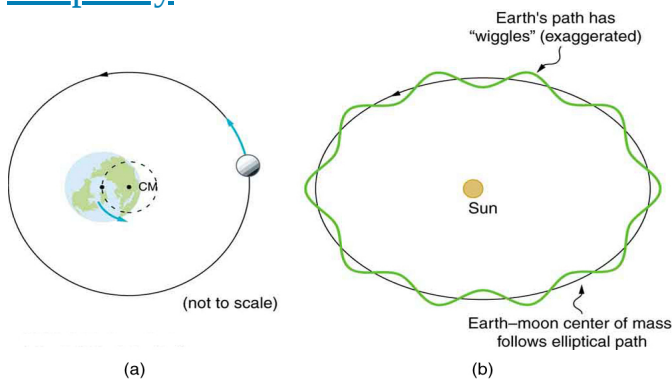


The direction of the acceleration is toward the center of the Earth.

### Discussion

The centripetal acceleration of the Moon found in (b) differs by less than 1% from the acceleration due to Earth's gravity found in (a). This agreement is approximate because the Moon's orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its center of mass, which is located some 1700 km below Earth's surface). The clear implication is that Earth's gravitational force causes the Moon to orbit Earth.

Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton's third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see [\[link\]](#)). We do not sense the Moon's effect on Earth's motion, because the Moon's gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon's gravitational force as discussed in [Satellites and Kepler's Laws: An Argument for Simplicity](#).

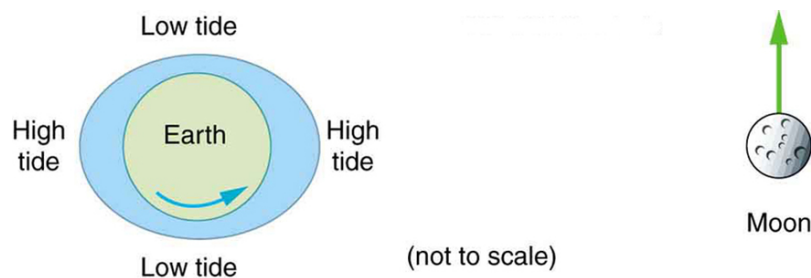


(a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth's path around the Sun has "wiggles" in it. Similar wiggles in the paths of stars have been observed and are

considered direct evidence of planets orbiting those stars. This is important because the planets' reflected light is often too dim to be observed.

## Tides

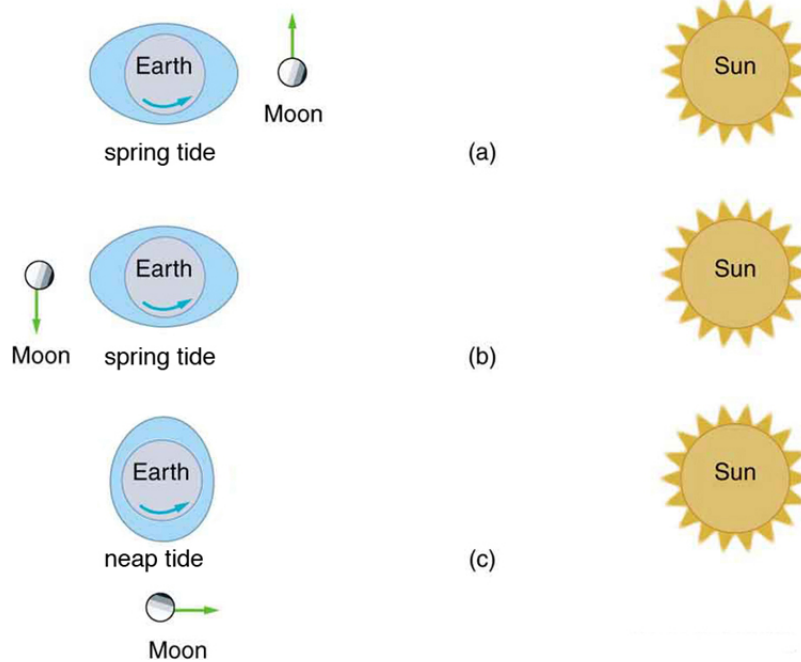
Ocean tides are one very observable result of the Moon's gravity acting on Earth. [\[link\]](#) is a simplified drawing of the Moon's position relative to the tides. Because water easily flows on Earth's surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon's gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well).



The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are

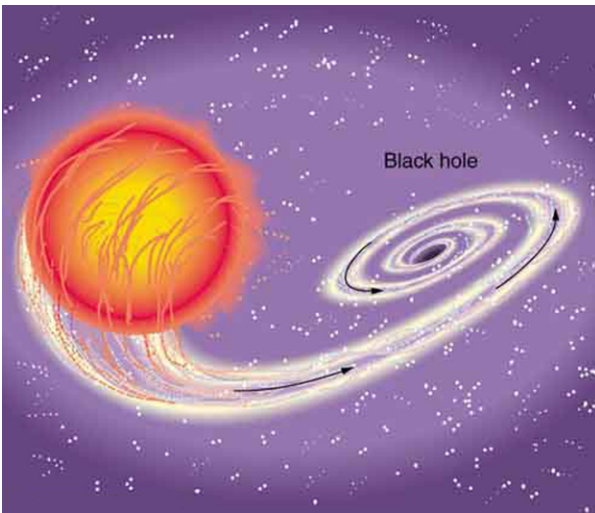
not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.

The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a  $90^\circ$  angle to the Earth-Moon alignment.



(a, b) Spring tides: The highest tides occur when Earth, the Moon, and the Sun are aligned. (c) Neap tide: The lowest tides occur when the Sun lies at  $90^\circ$  to the Earth-Moon alignment. Note that this figure is not drawn to scale.

Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see [\[link\]](#)). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers across. The tidal forces near them are so great that they can actually tear matter from a companion star.



A black hole is an object with such strong gravity that not even light can escape it. This black hole was created by the supernova of one star in a two-star system. The tidal forces created by the black hole are so great that it tears matter from the companion star. This matter is compressed and heated as it is sucked into the black hole, creating light and X-rays observable from Earth.

## **”Weightlessness” and Microgravity**

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What is the effect of “weightlessness” upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn’t mean that an astronaut is not being acted upon by the gravitational force. There is no “zero gravity” in an astronaut’s orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.



Astronauts experiencing weightlessness on board the International Space Station. (credit: NASA)

**Microgravity** refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

## The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant  $G$  is determined experimentally. This definition was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of  $G$  is very basic and important because it determines the strength of one of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in [\[link\]](#). Remarkably, his value for  $G$  differs by less than 1% from the best modern value.

One important consequence of knowing  $G$  was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth  $M$  from the relationship Newton's universal law of gravitation gives

**Equation:**

$$mg = G \frac{mM}{r^2},$$

where  $m$  is the mass of the object,  $M$  is the mass of Earth, and  $r$  is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See [\[link\]](#). The mass  $m$  of the object cancels, leaving an equation for  $g$ :

**Equation:**

$$g = G \frac{M}{r^2}.$$

Rearranging to solve for  $M$  yields

**Equation:**

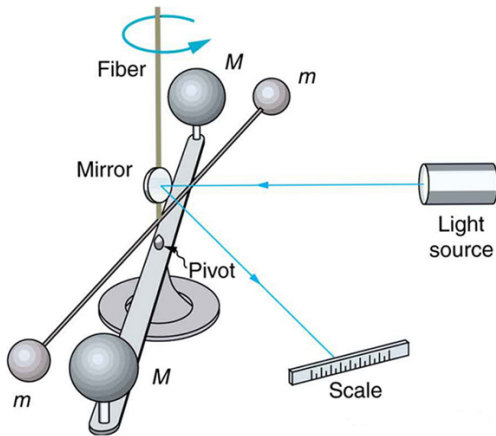
$$M = \frac{gr^2}{G}.$$

So  $M$  can be calculated because all quantities on the right, including the radius of Earth  $r$ , are known from direct measurements. We shall see in [Satellites and Kepler's Laws: An Argument for Simplicity](#) that knowing  $G$  also allows for the determination of astronomical masses. Interestingly, of all the fundamental constants in physics,  $G$  is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös' measurements.

Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton's law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.





Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres ( $m$ ) and the two on the stand ( $M$ ) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.

## Section Summary

- Newton's universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is

**Equation:**

$$F = G \frac{mM}{r^2},$$

where  $F$  is the magnitude of the gravitational force.  $G$  is the gravitational constant, given by  $G = 6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ .

- Newton's law of gravitation applies universally.

**Conceptual Questions****Exercise:****Problem:**

Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

**Exercise:****Problem:**

Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not  $9.80 \text{ m/s}^2$ . Who do you agree with and why?

**Exercise:****Problem:**

Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

**Exercise:**

**Problem:**

Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

**Problem Exercises****Exercise:****Problem:**

(a) Calculate Earth's mass given the acceleration due to gravity at the North Pole is  $9.830 \text{ m/s}^2$  and the radius of the Earth is 6371 km from center to pole.

(b) Compare this with the accepted value of  $5.979 \times 10^{24} \text{ kg}$ .

---

**Solution:**

a)  $5.979 \times 10^{24} \text{ kg}$

b) This is identical to the best value to three significant figures.

**Exercise:****Problem:**

(a) Calculate the magnitude of the acceleration due to gravity on the surface of Earth due to the Moon.

(b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.

(c) Take the ratio of the Moon's acceleration to the Sun's and comment on why the tides are predominantly due to the Moon in spite of this

number.

**Exercise:**

**Problem:**

- (a) What is the acceleration due to gravity on the surface of the Moon?
  - (b) On the surface of Mars? The mass of Mars is  $6.418 \times 10^{23}$  kg and its radius is  $3.38 \times 10^6$  m.
- 

**Solution:**

- a)  $1.62 \text{ m/s}^2$
- b)  $3.75 \text{ m/s}^2$

**Exercise:**

**Problem:**

- (a) Calculate the acceleration due to gravity on the surface of the Sun.
- (b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

**Exercise:**

**Problem:**

The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.)

- (a) Calculate the magnitude of the acceleration due to the Moon's gravity at that point.
- (b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a).

Comment on whether or not they are equal and why they should or should not be.

---

**Solution:**

a)  $3.42 \times 10^{-5} \text{ m/s}^2$

b)  $3.34 \times 10^{-5} \text{ m/s}^2$

The values are nearly identical. One would expect the gravitational force to be the same as the centripetal force at the core of the system.

**Exercise:**

**Problem:** Solve part (b) of [\[link\]](#) using  $a_c = v^2/r$ .

**Exercise:**

**Problem:**

Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one's birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).

(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some  $6.29 \times 10^{11} \text{ m}$  away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

---

**Solution:**

a)  $7.01 \times 10^{-7} \text{ N}$

b)  $1.35 \times 10^{-6} \text{ N}$ , 0.521

**Exercise:**

**Problem:**

The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune's orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune's orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:

(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are  $4.50 \times 10^{12} \text{ m}$  apart, as they are at present. The mass of Pluto is  $1.4 \times 10^{22} \text{ kg}$ .

(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about  $2.50 \times 10^{12} \text{ m}$  apart, and compare it with that due to Pluto. The mass of Uranus is  $8.62 \times 10^{25} \text{ kg}$ .

**Exercise:**

**Problem:**

(a) The Sun orbits the Milky Way galaxy once each  $2.60 \times 10^8 \text{ y}$ , with a roughly circular orbit averaging  $3.00 \times 10^4$  light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun?

(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?

---

**Solution:**

a)  $1.66 \times 10^{-10} \text{ m/s}^2$

b)  $2.17 \times 10^5 \text{ m/s}$

## Exercise:

### Problem: Unreasonable Result

A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.

- (a) Calculate the mass of the mountain.
  - (b) Compare the mountain's mass with that of Earth.
  - (c) What is unreasonable about these results?
  - (d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)
- 

### Solution:

a)  $2.937 \times 10^{17} \text{ kg}$

b)  $4.91 \times 10^{-8}$

of the Earth's mass.

c) The mass of the mountain and its fraction of the Earth's mass are too great.

d) The gravitational force assumed to be exerted by the mountain is too great.

## Glossary

gravitational constant,  $G$

a proportionality factor used in the equation for Newton's universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

center of mass

the point where the entire mass of an object can be thought to be concentrated

microgravity

an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton's universal law of gravitation

every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them



## Satellites and Kepler's Laws: An Argument for Simplicity

- State Kepler's laws of planetary motion.
- Derive the third Kepler's law for circular orbits.
- Discuss the Ptolemaic model of the universe.

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon's orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

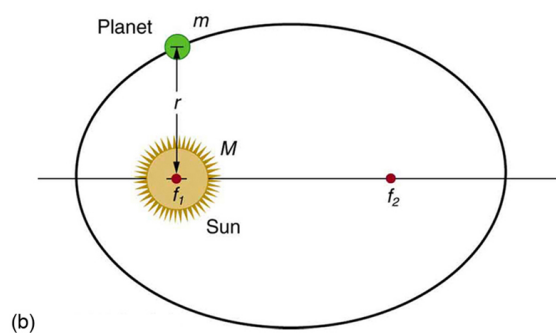
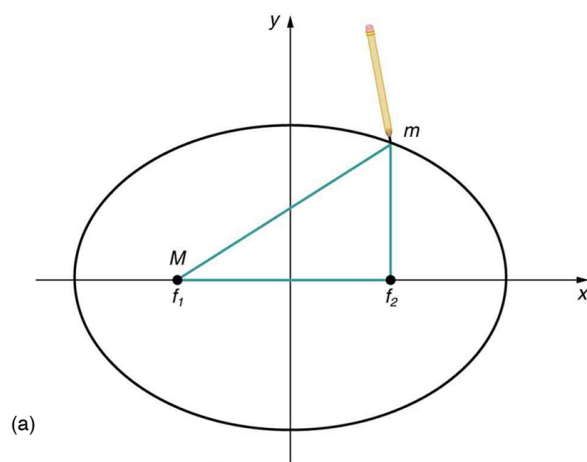
1. *A small mass  $m$  orbits a much larger mass  $M$ .* This allows us to view the motion as if  $M$  were stationary—in fact, as if from an inertial frame of reference placed on  $M$ —without significant error. Mass  $m$  is the satellite of  $M$ , if the orbit is gravitationally bound.
2. *The system is isolated from other masses.* This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth's satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler's laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

# Kepler's Laws of Planetary Motion

## Kepler's First Law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.



(a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci ( $f_1$  and  $f_2$ ) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the

two foci coincide (thus any point on the circle is the same distance from the center). (b)

For any closed gravitational orbit,  $m$  follows an elliptical path with  $M$  at one focus.

Kepler's first law states this fact for planets orbiting the Sun.

### Kepler's Second Law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see [\[link\]](#)).

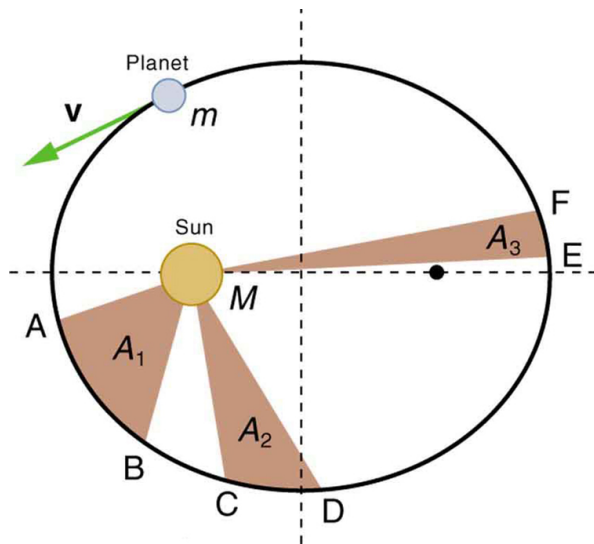
### Kepler's Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

**Equation:**

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where  $T$  is the period (time for one orbit) and  $r$  is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.



The shaded regions have equal areas. It takes equal times for  $m$  to go from A to B, from C to D, and from E to F. The mass  $m$  moves fastest when it is closest to  $M$ . Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.

Note again that while, for historical reasons, Kepler's laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

### Example:

#### Find the Time for One Orbit of an Earth Satellite

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of  $3.84 \times 10^8$  m from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

#### Strategy

The period, or time for one orbit, is related to the radius of the orbit by Kepler's third law, given in mathematical form in  $\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$ . Let us use the subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find  $T_2$ . The given information tells us that the orbital radius of the Moon is  $r_1 = 3.84 \times 10^8$  m, and that the period of the Moon is  $T_1 = 27.3$  d. The height of the artificial satellite above Earth's surface is given, and so we must add the radius of Earth (6380 km) to get  $r_2 = (1500 + 6380)$  km = 7880 km. Now all quantities are known, and so  $T_2$  can be found.

**Solution**

Kepler's third law is

**Equation:**

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

To solve for  $T_2$ , we cross-multiply and take the square root, yielding

**Equation:**

$$T_2^2 = T_1^2 \left( \frac{r_2}{r_1} \right)^3$$

**Equation:**

$$T_2 = T_1 \left( \frac{r_2}{r_1} \right)^{3/2}.$$

Substituting known values yields

**Equation:**

$$\begin{aligned} T_2 &= 27.3 \text{ d} \times \frac{24.0 \text{ h}}{\text{d}} \times \left( \frac{7880 \text{ km}}{3.84 \times 10^5 \text{ km}} \right)^{3/2} \\ &= 1.93 \text{ h.} \end{aligned}$$

**Discussion** This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the same amount

of time. This fact is related to the condition that the satellite's mass is small compared with that of Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler's, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover *what* was happening, Newton discovered that gravitational force was the cause.

### Derivation of Kepler's Third Law for Circular Orbits

We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler's laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass  $m$  around a large mass  $M$ , satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass  $m$ . Starting with Newton's second law applied to circular motion,

**Equation:**

$$F_{\text{net}} = ma_c = m \frac{v^2}{r}.$$

The net external force on mass  $m$  is gravity, and so we substitute the force of gravity for  $F_{\text{net}}$ :

**Equation:**

$$G \frac{mM}{r^2} = m \frac{v^2}{r}.$$

The mass  $m$  cancels, yielding

**Equation:**

$$G \frac{M}{r} = v^2.$$

The fact that  $m$  cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius  $r$ , all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler's third law, we must get the period  $T$  into the equation. By definition, period  $T$  is the time for one complete orbit. Now the average speed  $v$  is the circumference divided by the period—that is,

**Equation:**

$$v = \frac{2\pi r}{T}.$$

Substituting this into the previous equation gives

**Equation:**

$$G \frac{M}{r} = \frac{4\pi^2 r^2}{T^2}.$$

Solving for  $T^2$  yields

**Equation:**

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

**Equation:**

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

This is Kepler's third law. Note that Kepler's third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body  $M$  cancel.

Now consider what we get if we solve  $T^2 = \frac{4\pi^2}{GM} r^3$  for the ratio  $r^3/T^2$ . We obtain a relationship that can be used to determine the mass  $M$  of a parent body from the orbits of its satellites:

**Equation:**

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} M.$$

If  $r$  and  $T$  are known for a satellite, then the mass  $M$  of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio  $r^3/T^2$  should be a constant for all satellites of the same parent body (because  $r^3/T^2 = GM/4\pi^2$ ). (See [\[link\]](#)).

It is clear from [\[link\]](#) that the ratio of  $r^3/T^2$  is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the  $r$  and  $T$  data, and perturbations of the orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton's universal law of gravitation.

**Note:**

**Making Connections**

Newton's universal law of gravitation is modified by Einstein's general theory of relativity, as we shall see in [Particle Physics](#). Newton's gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein's modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.



---

## The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

- 1. is in orbit around the Sun,
- 2. has sufficient mass to assume hydrostatic equilibrium and
- 3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as “dwarf planet.”

In 2006, Pluto was demoted to a ‘dwarf planet’ after scientists revised their definition of what constitutes a “true” planet.

Parent	Satellite	Average orbital radius $r(\text{km})$	Period $T(\text{y})$	$r^3 / T^2 (\text{km}^3 / \text{y}^2)$
Earth	Moon	$3.84 \times 10^5$	0.07481	$1.01 \times 10^{19}$
Sun	Mercury	$5.79 \times 10^7$	0.2409	$3.34 \times 10^{24}$

---

Parent	Satellite	Average orbital radius $r(\text{km})$	Period $T(\text{y})$	$r^3 / T^2 (\text{km}^3 / \text{y}^2)$
	Venus	$1.082 \times 10^8$	0.6150	$3.35 \times 10^{24}$
	Earth	$1.496 \times 10^8$	1.000	$3.35 \times 10^{24}$
	Mars	$2.279 \times 10^8$	1.881	$3.35 \times 10^{24}$
	Jupiter	$7.783 \times 10^8$	11.86	$3.35 \times 10^{24}$
	Saturn	$1.427 \times 10^9$	29.46	$3.35 \times 10^{24}$
	Neptune	$4.497 \times 10^9$	164.8	$3.35 \times 10^{24}$
	Pluto	$5.90 \times 10^9$	248.3	$3.33 \times 10^{24}$
Jupiter	Io	$4.22 \times 10^5$	0.00485 (1.77 d)	$3.19 \times 10^{21}$
	Europa	$6.71 \times 10^5$	0.00972 (3.55 d)	$3.20 \times 10^{21}$

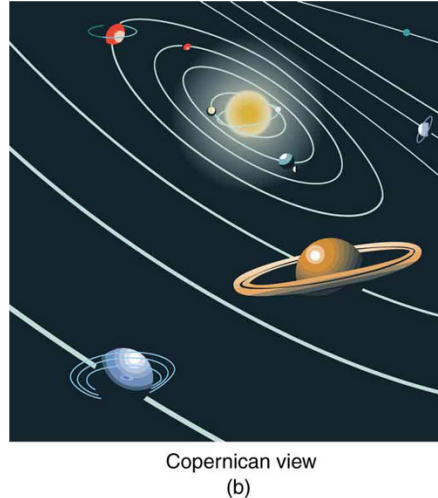
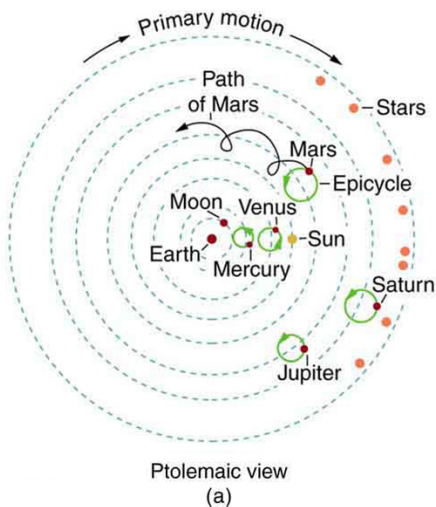
Parent	Satellite	Average orbital radius $r(\text{km})$	Period $T(\text{y})$	$r^3 / T^2 (\text{km}^3 / \text{y}^2)$
	Ganymede	$1.07 \times 10^6$	0.0196 (7.16 d)	$3.19 \times 10^{21}$
	Callisto	$1.88 \times 10^6$	0.0457 (16.19 d)	$3.20 \times 10^{21}$

### Orbital Data and Kepler's Third Law

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in [\[link\]](#)(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

[\[link\]](#)(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.



(a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.

## Section Summary

- Kepler's laws are stated for a small mass  $m$  orbiting a larger mass  $M$  in near-isolation. Kepler's laws of planetary motion are then as follows:

Kepler's first law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

Kepler's second law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

Kepler's third law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

**Equation:**

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where  $T$  is the period (time for one orbit) and  $r$  is the average radius of the orbit.

- The period and radius of a satellite's orbit about a larger body  $M$  are related by

**Equation:**

$$T^2 = \frac{4\pi^2}{GM} r^3$$

or

**Equation:**

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} M.$$

## Conceptual Questions

**Exercise:**

**Problem:**

In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?

## Problem Exercises

### Exercise:

#### Problem:

A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth's rotation). Calculate the radius of such an orbit based on the data for the moon in [\[link\]](#).

### Exercise:

#### Problem:

Calculate the mass of the Sun based on data for Earth's orbit and compare the value obtained with the Sun's actual mass.

---

#### Solution:

$$1.98 \times 10^{30} \text{ kg}$$

### Exercise:

#### Problem:

Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

### Exercise:

#### Problem:

Find the ratio of the mass of Jupiter to that of Earth based on data in [\[link\]](#).

---

#### Solution:

$$\frac{M_J}{M_E} = 316$$

### Exercise:

**Problem:**

Astronomical observations of our Milky Way galaxy indicate that it has a mass of about  $8.0 \times 10^{11}$  solar masses. A star orbiting on the galaxy's periphery is about  $6.0 \times 10^4$  light years from its center. (a) What should the orbital period of that star be? (b) If its period is  $6.0 \times 10^7$  years instead, what is the mass of the galaxy? Such calculations are used to imply the existence of "dark matter" in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

**Exercise:****Problem: Integrated Concepts**

Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth's surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite's orbit at an angle of  $90^\circ$  relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite's velocity does not change appreciably, because its mass is much greater than the rivet's.)

---

**Solution:**

a)  $7.4 \times 10^3 \text{ m/s}$

b)  $1.05 \times 10^3 \text{ m/s}$

c)  $2.86 \times 10^{-7} \text{ s}$

d)  $1.84 \times 10^7 \text{ N}$

e)  $2.76 \times 10^4 \text{ J}$

**Exercise:**

**Problem: Unreasonable Results**

(a) Based on Kepler's laws and information on the orbital characteristics of the Moon, calculate the orbital radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

---

**Solution:**

a)  $5.08 \times 10^3 \text{ km}$

b) This radius is unreasonable because it is less than the radius of earth.

c) The premise of a one-hour orbit is inconsistent with the known radius of the earth.

**Exercise:****Problem: Construct Your Own Problem**

On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.



## Introduction to Work, Energy, and Energy Resources

class="introduction"

How many  
forms of  
energy can  
you identify  
in this  
photograph  
of a wind  
farm in  
Iowa?  
(credit:  
Jürgen from  
Sandesneben  
, Germany,  
Wikimedia  
Commons)



*Energy* plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods, to the energy we use to run our cars, to the sunlight that warms us on the beach. You can also cite examples of what people call energy that may not be scientific, such as someone having an energetic personality. Not only does energy have many interesting forms, it is

involved in almost all phenomena, and is one of the most important concepts of physics. What makes it even more important is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is *conserved*.

**Conservation of energy** (as physicists like to call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he suggested that mass is equivalent to energy (his famous equation  $E = mc^2$ ).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally. We will briefly examine the world's energy use patterns at the end of this chapter.

There is no simple, yet accurate, scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define **energy** as the ability to do work, admitting that in some circumstances not all energy is available to do work. Because of the association of energy with work, we begin the chapter with a discussion of work. Work is intimately related to energy and how energy moves from one system to another or changes form.

## Work: The Scientific Definition

- Explain how an object must be displaced for a force on it to do work.
- Explain how relative directions of force and displacement determine whether the work done is positive, negative, or zero.

## What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done, a force must be exerted and there must be displacement in the direction of the force.

Formally, the **work** done on a system by a constant force is defined to be *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

**Equation:**

$$W = | \mathbf{F} | (\cos \theta) | \mathbf{d} |,$$

where  $W$  is work,  $\mathbf{d}$  is the displacement of the system, and  $\theta$  is the angle between the force vector  $\mathbf{F}$  and the displacement vector  $\mathbf{d}$ , as in [\[link\]](#). We can also write this as

**Equation:**

$$W = Fd \cos \theta.$$

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.

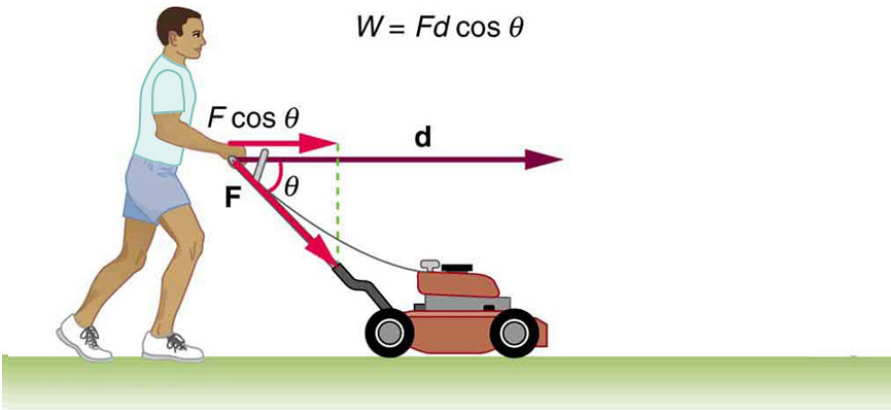
**Note:****What is Work?**

The work done on a system by a constant force is *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

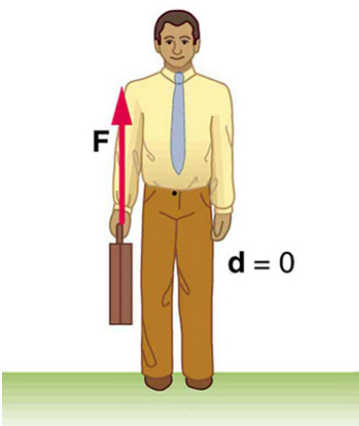
**Equation:**

$$W = Fd \cos \theta,$$

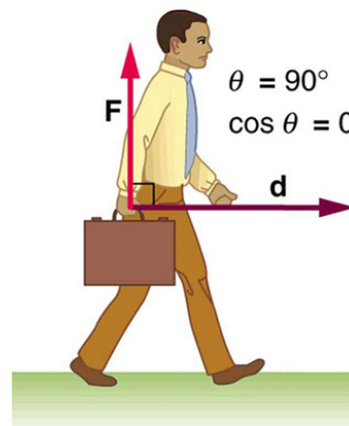
where  $W$  is work,  $F$  is the magnitude of the force on the system,  $d$  is the magnitude of the displacement of the system, and  $\theta$  is the angle between the force vector  $\mathbf{F}$  and the displacement vector  $\mathbf{d}$ .



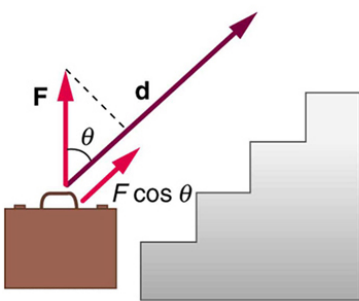
(a)



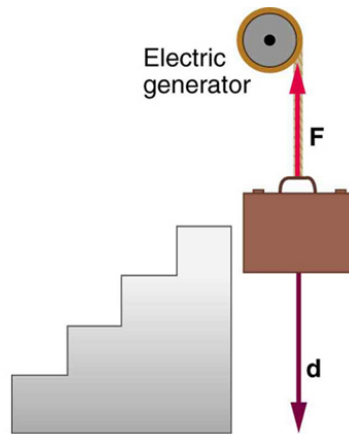
(b)



(c)



(d)



(e)

Examples of work. (a) The work done by the force  $\mathbf{F}$  on this lawn mower is  $Fd \cos \theta$ . Note that  $F \cos \theta$  is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no

displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work is done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force  $\mathbf{F}$  in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because  $\mathbf{F}$  and  $\mathbf{d}$  are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in [\[link\]](#). The person holding the briefcase in [\[link\]\(b\)](#) does no work, for example. Here  $d = 0$ , so  $W = 0$ . Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, *but they are doing no work on the system of interest* (the “briefcase-Earth system”—see [Gravitational Potential Energy](#) for more details). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase on level ground in [\[link\]\(c\)](#) does no work on it, because the force is perpendicular to the motion. That is,  $\cos 90^\circ = 0$ , and so  $W = 0$ .

In contrast, when a force exerted on the system has a component in the direction of motion, such as in [\[link\]\(d\)](#), work is done—energy is transferred to the briefcase. Finally, in [\[link\]\(e\)](#), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase’s weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward

on the briefcase, and the displacement downward. This makes  $\theta = 180^\circ$ , and  $\cos 180^\circ = -1$ ; therefore,  $W$  is negative.

## Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in **newton-meters**. A newton-meter is given the special name **joule** (J), and  $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$ . One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

### Example:

#### Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in [\[link\]](#)(a) if he exerts a constant force of 75.0 N at an angle  $35^\circ$  below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of 10,000 kJ (about 2400 kcal) of food energy. One *calorie* (1 cal) of heat is the amount required to warm 1 g of water by  $1^\circ\text{C}$ , and is equivalent to 4.184 J, while one *food calorie* (1 kcal) is equivalent to 4184 J.

#### Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation  $W = Fd \cos \theta$ . The force, angle, and displacement are given, so that only the work  $W$  is unknown.

#### Solution

The equation for the work is

#### Equation:

$$W = Fd \cos \theta.$$

Substituting the known values gives

**Equation:**

$$\begin{aligned} W &= (75.0 \text{ N})(25.0 \text{ m}) \cos (35.0^\circ) \\ &= 1536 \text{ J} = 1.54 \times 10^3 \text{ J}. \end{aligned}$$

Converting the work in joules to kilocalories yields

$W = (1536 \text{ J})(1 \text{ kcal}/4184 \text{ J}) = 0.367 \text{ kcal}$ . The ratio of the work done to the daily consumption is

**Equation:**

$$\frac{W}{2400 \text{ kcal}} = 1.53 \times 10^{-4}.$$

**Discussion**

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we “work” all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

**Section Summary**

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work  $W$  that a force  $\mathbf{F}$  does on an object is the product of the magnitude  $F$  of the force, times the magnitude  $d$  of the displacement, times the cosine of the angle  $\theta$  between them. In symbols,

**Equation:**

$$W = Fd \cos \theta.$$

- The SI unit for work and energy is the joule (J), where  $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$ .
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.



- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.

## Conceptual Questions

### Exercise:

#### Problem:

Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

### Exercise:

#### Problem:

Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

### Exercise:

#### Problem:

Describe a situation in which a force is exerted for a long time but does no work. Explain.

## Problems & Exercises

### Exercise:

#### Problem:

How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

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#### Solution:

**Equation:**

$$3.00 \text{ J} = 7.17 \times 10^{-4} \text{ kcal}$$

**Exercise:****Problem:**

A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

**Exercise:****Problem:**

(a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

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**Solution:**

(a)  $5.92 \times 10^5 \text{ J}$

(b)  $-5.88 \times 10^5 \text{ J}$

(c) The net force is zero.

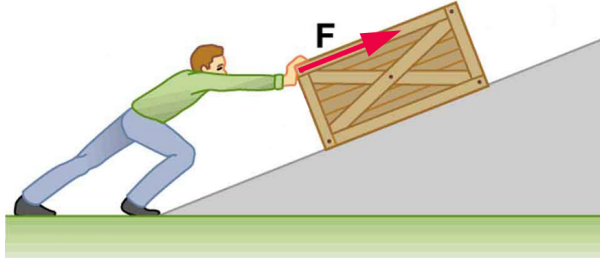
**Exercise:****Problem:**

Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See [\[link\]](#) for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

**Exercise:**

**Problem:**

Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of  $20.0^\circ$  with the horizontal. (See [\[link\]](#).) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate *and* on his body to get up the ramp.



A man pushes a crate up a ramp.

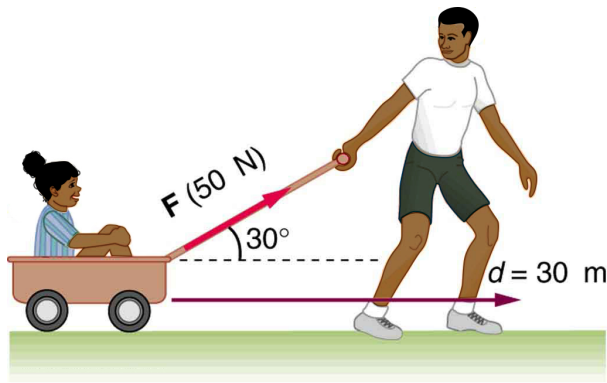
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**Solution:****Equation:**

$$3.14 \times 10^3 \text{ J}$$

**Exercise:****Problem:**

How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in [\[link\]](#)? Assume no friction acts on the wagon.



The boy does work on the system of the wagon and the child when he pulls them as shown.

**Exercise:**

**Problem:**

A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction  $25.0^\circ$  below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

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**Solution:**

(a)  $-700 \text{ J}$

(b)  $0$

(c)  $700 \text{ J}$

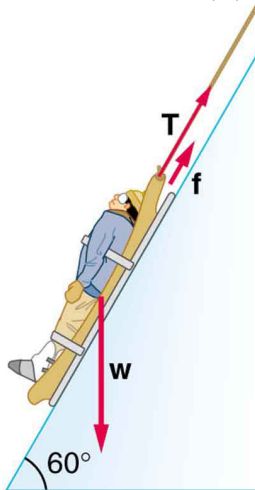
(d)  $38.6 \text{ N}$

(e) 0

### Exercise:

#### Problem:

Suppose the ski patrol lowers a rescue sled and victim, having a total mass of  $90.0\text{ kg}$ , down a  $60.0^\circ$  slope at constant speed, as shown in [\[link\]](#). The coefficient of friction between the sled and the snow is  $0.100$ . (a) How much work is done by friction as the sled moves  $30.0\text{ m}$  along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?



A rescue sled and victim are lowered down a steep slope.

### Glossary

energy

the ability to do work

work

the transfer of energy by a force that causes an object to be displaced;  
the product of the component of the force in the direction of the  
displacement and the magnitude of the displacement

joule

SI unit of work and energy, equal to one newton-meter

## Kinetic Energy and the Work-Energy Theorem

- Explain work as a transfer of energy and net work as the work done by the net force.
- Explain and apply the work-energy theorem.

## Work Transfers Energy

What happens to the work done on a system? Energy is transferred into the system, but in what form? Does it remain in the system or move on? The answers depend on the situation. For example, if the lawn mower in [\[link\]](#) (a) is pushed just hard enough to keep it going at a constant speed, then energy put into the mower by the person is removed continuously by friction, and eventually leaves the system in the form of heat transfer. In contrast, work done on the briefcase by the person carrying it up stairs in [\[link\]](#) (d) is stored in the briefcase-Earth system and can be recovered at any time, as shown in [\[link\]](#) (e). In fact, the building of the pyramids in ancient Egypt is an example of storing energy in a system by doing work on the system. Some of the energy imparted to the stone blocks in lifting them during construction of the pyramids remains in the stone-Earth system and has the potential to do work.

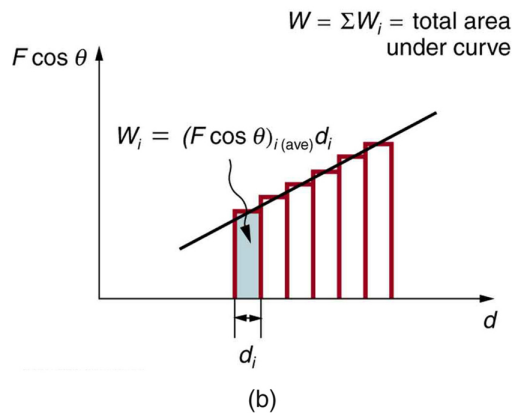
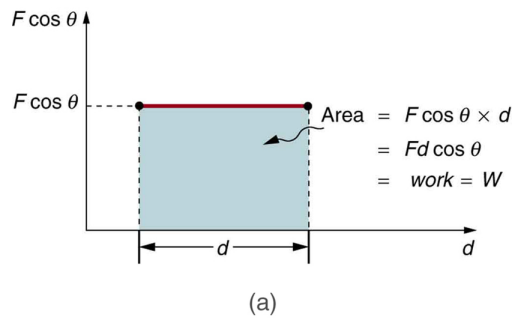
In this section we begin the study of various types of work and forms of energy. We will find that some types of work leave the energy of a system constant, for example, whereas others change the system in some way, such as making it move. We will also develop definitions of important forms of energy, such as the energy of motion.

## Net Work and the Work-Energy Theorem

We know from the study of Newton's laws in [Dynamics: Force and Newton's Laws of Motion](#) that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work done by all external forces—that is, **net work** is the work done by the net external force  $\mathbf{F}_{\text{net}}$ . In equation form, this is  $W_{\text{net}} = F_{\text{net}}d \cos \theta$  where  $\theta$  is the angle between the force vector and the displacement vector.

[\[link\]](#)(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an  $F \cos \theta$  vs.  $d$  graph. In this case,  $F \cos \theta$  is constant. You can see that the area under the graph is  $Fd \cos \theta$ , or the work done. [\[link\]](#)(b) shows a more general process where the force varies. The area under the curve is divided into strips, each having an average force  $(F \cos \theta)_{i(\text{ave})}$ . The work done is  $(F \cos \theta)_{i(\text{ave})}d_i$  for each strip, and the total work done is the sum of the  $W_i$ . Thus the total work done is the total area under the curve, a useful property to which we shall refer later.



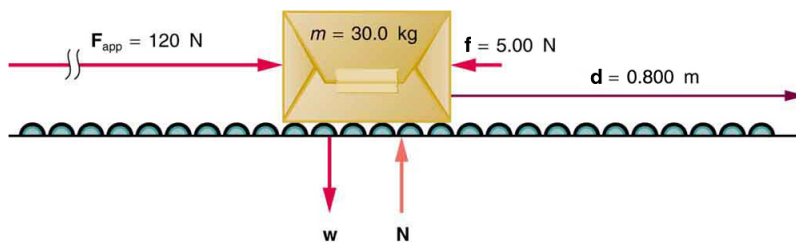
(a) A graph of  $F \cos \theta$  vs.  $d$ , when  $F \cos \theta$  is



constant. The area under the curve represents the work done by the force.

(b) A graph of  $F \cos \theta$  vs.  $d$  in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.

Net work will be simpler to examine if we consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in [\[link\]](#).



A package on a roller belt is pushed horizontally through a distance  $d$ .

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force arises solely from the horizontal applied force  $F_{\text{app}}$  and the horizontal friction force  $f$ . Thus, as expected, the net force is

parallel to the displacement, so that  $\theta = 0^\circ$  and  $\cos \theta = 1$ , and the net work is given by

**Equation:**

$$W_{\text{net}} = F_{\text{net}}d.$$

The effect of the net force  $\mathbf{F}_{\text{net}}$  is to accelerate the package from  $v_0$  to  $v$ . The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See [\[link\]](#).) By using Newton's second law, and doing some algebra, we can reach an interesting conclusion. Substituting  $F_{\text{net}} = ma$  from Newton's second law gives

**Equation:**

$$W_{\text{net}} = mad.$$

To get a relationship between net work and the speed given to a system by the net force acting on it, we take  $d = x - x_0$  and use the equation studied in [Motion Equations for Constant Acceleration in One Dimension](#) for the change in speed over a distance  $d$  if the acceleration has the constant value  $a$ ; namely,  $v^2 = v_0^2 + 2ad$  (note that  $a$  appears in the expression for the net work). Solving for acceleration gives  $a = \frac{v^2 - v_0^2}{2d}$ . When  $a$  is substituted into the preceding expression for  $W_{\text{net}}$ , we obtain

**Equation:**

$$W_{\text{net}} = m \left( \frac{v^2 - v_0^2}{2d} \right) d.$$

The  $d$  cancels, and we rearrange this to obtain

**Equation:**

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

This expression is called the **work-energy theorem**, and it actually applies *in general* (even for forces that vary in direction and magnitude), although we have derived it for the special case of a constant force parallel to the displacement. The theorem implies that the net work on a system equals the change in the quantity  $\frac{1}{2}mv^2$ . This quantity is our first example of a form of energy.

**Note:**

**The Work-Energy Theorem**

The net work on a system equals the change in the quantity  $\frac{1}{2}mv^2$ .

**Equation:**

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

The quantity  $\frac{1}{2}mv^2$  in the work-energy theorem is defined to be the translational **kinetic energy** (KE) of a mass  $m$  moving at a speed  $v$ . (*Translational* kinetic energy is distinct from *rotational* kinetic energy, which is considered later.) In equation form, the translational kinetic energy,

**Equation:**

$$\text{KE} = \frac{1}{2}mv^2,$$

is the energy associated with translational motion. Kinetic energy is a form of energy associated with the motion of a particle, single body, or system of objects moving together.

We are aware that it takes energy to get an object, like a car or the package in [\[link\]](#), up to speed, but it may be a bit surprising that kinetic energy is proportional to speed squared. This proportionality means, for example, that a car traveling at 100 km/h has four times the kinetic energy it has at 50

km/h, helping to explain why high-speed collisions are so devastating. We will now consider a series of examples to illustrate various aspects of work and energy.

**Example:**

**Calculating the Kinetic Energy of a Package**

Suppose a 30.0-kg package on the roller belt conveyor system in [\[link\]](#) is moving at 0.500 m/s. What is its kinetic energy?

**Strategy**

Because the mass  $m$  and speed  $v$  are given, the kinetic energy can be calculated from its definition as given in the equation  $\text{KE} = \frac{1}{2}mv^2$ .

**Solution**

The kinetic energy is given by

**Equation:**

$$\text{KE} = \frac{1}{2}mv^2.$$

Entering known values gives

**Equation:**

$$\text{KE} = 0.5(30.0 \text{ kg})(0.500 \text{ m/s})^2,$$

which yields

**Equation:**

$$\text{KE} = 3.75 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 3.75 \text{ J}.$$

**Discussion**

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

**Example:****Determining the Work to Accelerate a Package**

Suppose that you push on the 30.0-kg package in [\[link\]](#) with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

**Strategy and Concept for (a)**

This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. (See [\[link\]](#).) As expected, the net work is the net force times distance.

**Solution for (a)**

The net force is the push force minus friction, or

$F_{\text{net}} = 120 \text{ N} - 5.00 \text{ N} = 115 \text{ N}$ . Thus the net work is

**Equation:**

$$\begin{aligned} W_{\text{net}} &= F_{\text{net}}d = (115 \text{ N})(0.800 \text{ m}) \\ &= 92.0 \text{ N} \cdot \text{m} = 92.0 \text{ J.} \end{aligned}$$

**Discussion for (a)**

This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy. The net work equals the sum of the work done by each individual force.

**Strategy and Concept for (b)**

The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

**Solution for (b)**

The applied force does work.

**Equation:**

$$\begin{aligned}
 W_{\text{app}} &= F_{\text{app}} d \cos(0^\circ) = F_{\text{app}} d \\
 &= (120 \text{ N})(0.800 \text{ m}) \\
 &= 96.0 \text{ J}
 \end{aligned}$$

The friction force and displacement are in opposite directions, so that  $\theta = 180^\circ$ , and the work done by friction is

**Equation:**

$$\begin{aligned}
 W_{\text{fr}} &= F_{\text{fr}} d \cos(180^\circ) = -F_{\text{fr}} d \\
 &= -(5.00 \text{ N})(0.800 \text{ m}) \\
 &= -4.00 \text{ J.}
 \end{aligned}$$

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

**Equation:**

$$\begin{aligned}
 W_{\text{gr}} &= 0, \\
 W_{\text{N}} &= 0, \\
 W_{\text{app}} &= 96.0 \text{ J}, \\
 W_{\text{fr}} &= -4.00 \text{ J.}
 \end{aligned}$$

The total work done as the sum of the work done by each force is then seen to be

**Equation:**

$$W_{\text{total}} = W_{\text{gr}} + W_{\text{N}} + W_{\text{app}} + W_{\text{fr}} = 92.0 \text{ J.}$$

### Discussion for (b)

The calculated total work  $W_{\text{total}}$  as the sum of the work by each force agrees, as expected, with the work  $W_{\text{net}}$  done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

**Example:**

### Determining Speed from Work and Energy

Find the speed of the package in [\[link\]](#) at the end of the push, using work and energy concepts.

#### Strategy

Here the work-energy theorem can be used, because we have just calculated the net work,  $W_{\text{net}}$ , and the initial kinetic energy,  $\frac{1}{2}mv_0^2$ . These calculations allow us to find the final kinetic energy,  $\frac{1}{2}mv^2$ , and thus the final speed  $v$ .

#### Solution

The work-energy theorem in equation form is

#### Equation:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Solving for  $\frac{1}{2}mv^2$  gives

#### Equation:

$$\frac{1}{2}mv^2 = W_{\text{net}} + \frac{1}{2}mv_0^2.$$

Thus,

#### Equation:

$$\frac{1}{2}mv^2 = 92.0 \text{ J} + 3.75 \text{ J} = 95.75 \text{ J}.$$

Solving for the final speed as requested and entering known values gives

#### Equation:

$$\begin{aligned} v &= \sqrt{\frac{2(95.75 \text{ J})}{m}} = \sqrt{\frac{191.5 \text{ kg}\cdot\text{m}^2/\text{s}^2}{30.0 \text{ kg}}} \\ &= 2.53 \text{ m/s}. \end{aligned}$$

#### Discussion

Using work and energy, we not only arrive at an answer, we see that the final kinetic energy is the sum of the initial kinetic energy and the net work

done on the package. This means that the work indeed adds to the energy of the package.

**Example:**

**Work and Energy Can Reveal Distance, Too**

How far does the package in [\[link\]](#) coast after the push, assuming friction remains constant? Use work and energy considerations.

**Strategy**

We know that once the person stops pushing, friction will bring the package to rest. In terms of energy, friction does negative work until it has removed all of the package's kinetic energy. The work done by friction is the force of friction times the distance traveled times the cosine of the angle between the friction force and displacement; hence, this gives us a way of finding the distance traveled after the person stops pushing.

**Solution**

The normal force and force of gravity cancel in calculating the net force. The horizontal friction force is then the net force, and it acts opposite to the displacement, so  $\theta = 180^\circ$ . To reduce the kinetic energy of the package to zero, the work  $W_{\text{fr}}$  by friction must be minus the kinetic energy that the package started with plus what the package accumulated due to the pushing. Thus  $W_{\text{fr}} = -95.75 \text{ J}$ . Furthermore,  $W_{\text{fr}} = f d' \cos \theta = -f d'$ , where  $d'$  is the distance it takes to stop. Thus,

**Equation:**

$$d' = -\frac{W_{\text{fr}}}{f} = -\frac{-95.75 \text{ J}}{5.00 \text{ N}},$$

and so

**Equation:**

$$d' = 19.2 \text{ m}.$$

**Discussion**

This is a reasonable distance for a package to coast on a relatively friction-free conveyor system. Note that the work done by friction is negative (the



force is in the opposite direction of motion), so it removes the kinetic energy.

Some of the examples in this section can be solved without considering energy, but at the expense of missing out on gaining insights about what work and energy are doing in this situation. On the whole, solutions involving energy are generally shorter and easier than those using kinematics and dynamics alone.

## Section Summary

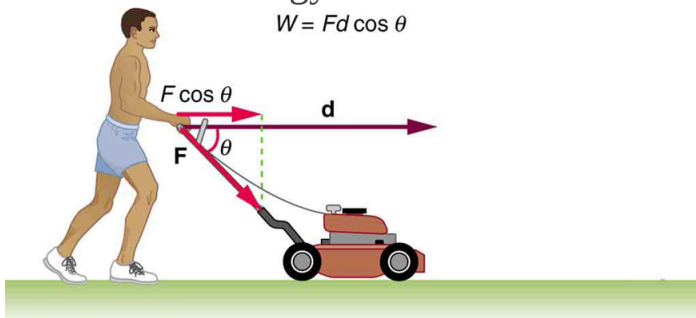
- The net work  $W_{\text{net}}$  is the work done by the net force acting on an object.
- Work done on an object transfers energy to the object.
- The translational kinetic energy of an object of mass  $m$  moving at speed  $v$  is  $\text{KE} = \frac{1}{2}mv^2$ .
- The work-energy theorem states that the net work  $W_{\text{net}}$  on a system changes its kinetic energy,  $W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$ .

## Conceptual Questions

### Exercise:

#### Problem:

The person in [\[link\]](#) does work on the lawn mower. Under what conditions would the mower gain energy? Under what conditions would it lose energy?



**Exercise:****Problem:**

Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

**Exercise:****Problem:**

When solving for speed in [\[link\]](#), we kept only the positive root. Why?

**Problems & Exercises****Exercise:****Problem:**

Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

---

**Solution:**

1/250

**Exercise:****Problem:**

(a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

**Exercise:****Problem:**

Confirm the value given for the kinetic energy of an aircraft carrier in [\[link\]](#). You will need to look up the definition of a nautical mile (1 knot = 1 nautical mile/h).

---

**Solution:**

$$1.1 \times 10^{10} \text{ J}$$

**Exercise:****Problem:**

(a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

**Exercise:****Problem:**

A car's bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.

---

**Solution:**

$$2.8 \times 10^3 \text{ N}$$

**Exercise:**

**Problem:**

Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent's face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the gory old days when no gloves were used and the knuckles and face would compress only 2.00 cm. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

**Exercise:****Problem:**

Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

---

**Solution:**

102 N

**Glossary**

net work

work done by the net force, or vector sum of all the forces, acting on an object

work-energy theorem

the result, based on Newton's laws, that the net work done on an object is equal to its change in kinetic energy

kinetic energy

the energy an object has by reason of its motion, equal to  $\frac{1}{2}mv^2$  for the translational (i.e., non-rotational) motion of an object of mass  $m$  moving at speed  $v$

## Gravitational Potential Energy

- Explain gravitational potential energy in terms of work done against gravity.
- Show that the gravitational potential energy of an object of mass  $m$  at height  $h$  on Earth is given by  $PE_g = mgh$ .
- Show how knowledge of the potential energy as a function of position can be used to simplify calculations and explain physical phenomena.

## Work Done Against Gravity

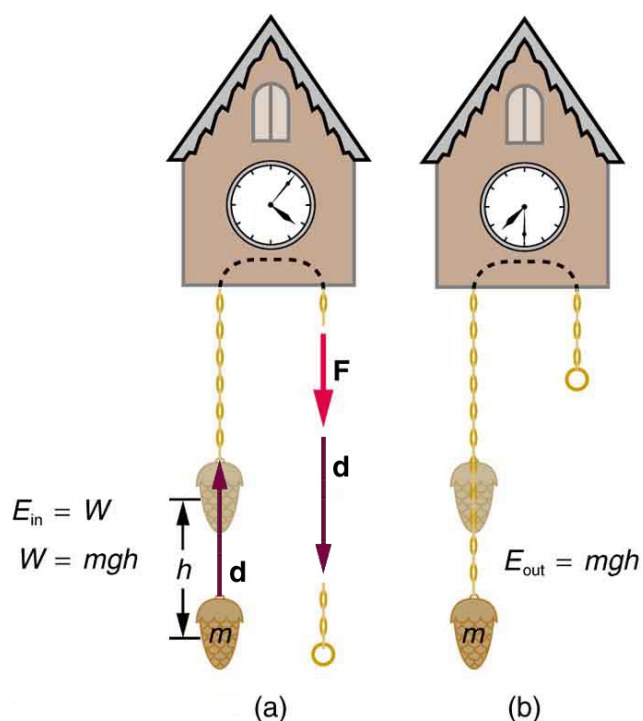
Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass  $m$  through a height  $h$ , such as in [\[link\]](#). If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight  $mg$ . The work done on the mass is then  $W = Fd = mgh$ . We define this to be the **gravitational potential energy** ( $PE_g$ ) put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the  $PE_g$  gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word “system”? Potential energy is a property of a system rather than of a single object—due to its physical position. An object’s gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth’s surface, but this point is arbitrary; what is important is the *difference* in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

## Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work

equal to  $mgh$  on it, thereby increasing its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of  $PE_g$  to KE without explicitly considering the intermediate step of work. (See [\[link\]](#).) This shortcut makes it is easier to solve problems using energy (if possible) rather than explicitly using forces.



(a) The work done to lift the weight is stored in the mass-Earth system as gravitational potential energy. (b) As the weight moves downward, this gravitational potential energy is transferred to the cuckoo clock.

More precisely, we define the *change* in gravitational potential energy  $\Delta PE_g$  to be  
**Equation:**

$$\Delta PE_g = mgh,$$

where, for simplicity, we denote the change in height by  $h$  rather than the usual  $\Delta h$ . Note that  $h$  is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

**Equation:**

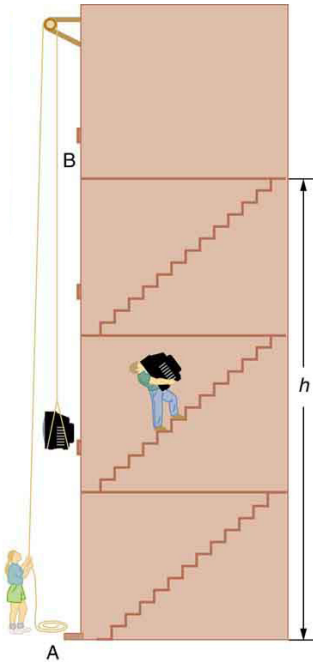
$$\begin{aligned} mgh &= (0.500 \text{ kg}) (9.80 \text{ m/s}^2) (1.00 \text{ m}) \\ &= 4.90 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 4.90 \text{ J}. \end{aligned}$$

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, *without directly considering the force of gravity that does the work*.

## Using Potential Energy to Simplify Calculations

The equation  $\Delta \text{PE}_g = mgh$  applies for any path that has a change in height of  $h$ , not just when the mass is lifted straight up. (See [\[link\]](#).) It is much easier to calculate  $mgh$  (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position  $h$  of a mass  $m$  is accompanied by a change in gravitational potential energy  $mgh$ , and we will avoid the equivalent but more difficult task of calculating work done by or against the gravitational force.





The change in  
gravitational  
potential energy  
( $\Delta PE_g$ )  
between points  
A and B is  
independent of  
the path.

$\Delta PE_g = mgh$   
for any path  
between the two  
points. Gravity  
is one of a small  
class of forces  
where the work  
done by or  
against the force  
depends only on  
the starting and  
ending points,  
not on the path  
between them.

**Example:****The Force to Stop Falling**

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

**Strategy**

This person's energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial  $PE_g$  is transformed into KE as he falls. The work done by the floor reduces this kinetic energy to zero.

**Solution**

The work done on the person by the floor as he stops is given by

**Equation:**

$$W = Fd \cos \theta = -Fd,$$

with a minus sign because the displacement while stopping and the force from floor are in opposite directions ( $\cos \theta = \cos 180^\circ = -1$ ). The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height  $h$ :

**Equation:**

$$KE = -\Delta PE_g = -mgh,$$

The distance  $d$  that the person's knees bend is much smaller than the height  $h$  of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work  $W$  done by the floor on the person stops the person and brings the person's kinetic energy to zero:

**Equation:**

$$W = -KE = mgh.$$

Combining this equation with the expression for  $W$  gives

**Equation:**

$$-Fd = mgh.$$

Recalling that  $h$  is negative because the person fell *down*, the force on the knee joints is given by

**Equation:**

$$F = -\frac{mgh}{d} = -\frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(-3.00 \text{ m})}{5.00 \times 10^{-3} \text{ m}} = 3.53 \times 10^5 \text{ N}.$$

### Discussion

Such a large force (500 times more than the person's weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo's hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump. (See [\[link\]](#).)

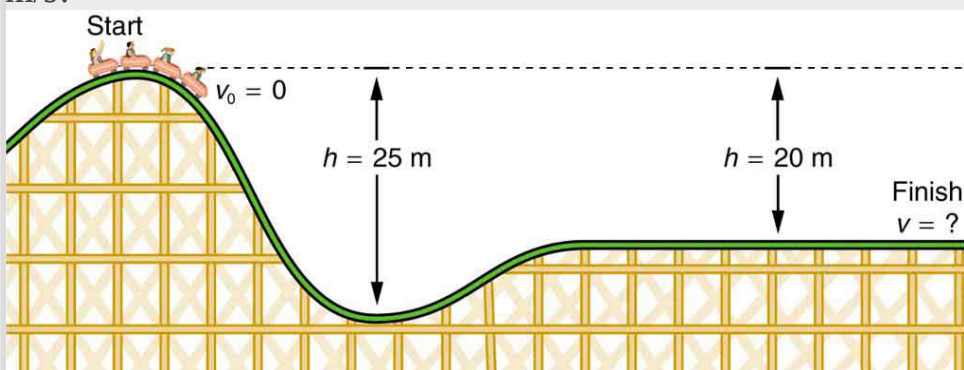


The work done by the ground upon the kangaroo reduces its kinetic energy to zero as it lands. However, by applying the force of the ground on the hind legs over a longer distance, the impact on the bones is reduced.  
(credit: Chris Samuel, Flickr)

### Example:

#### Finding the Speed of a Roller Coaster from its Height

(a) What is the final speed of the roller coaster shown in [\[link\]](#) if it starts from rest at the top of the 20.0 m hill and work done by frictional forces is negligible? (b) What is its final speed (again assuming negligible friction) if its initial speed is 5.00 m/s?



The speed of a roller coaster increases as gravity pulls it downhill and is greatest at its lowest point. Viewed in terms of energy, the roller-coaster-Earth system's gravitational potential energy is converted to kinetic energy. If work done by friction is negligible, all  $\Delta PE_g$  is converted to KE.

### Strategy

The roller coaster loses potential energy as it goes downhill. We neglect friction, so that the remaining force exerted by the track is the normal force, which is perpendicular to the direction of motion and does no work. The net work on the roller coaster is then done by gravity alone. The *loss* of gravitational potential energy from moving *downward* through a distance  $h$  equals the *gain* in kinetic energy. This can be written in equation form as  $-\Delta PE_g = \Delta KE$ . Using the equations for  $PE_g$  and KE, we can solve for the final speed  $v$ , which is the desired quantity.

### Solution for (a)

Here the initial kinetic energy is zero, so that  $\Delta KE = \frac{1}{2}mv^2$ . The equation for change in potential energy states that  $\Delta PE_g = mgh$ . Since  $h$  is negative in this case, we will rewrite this as  $\Delta PE_g = -mg |h|$  to show the minus sign clearly. Thus,

**Equation:**

$$-\Delta PE_g = \Delta KE$$

becomes

**Equation:**

$$mg | h | = \frac{1}{2}mv^2.$$

Solving for  $v$ , we find that mass cancels and that

**Equation:**

$$v = \sqrt{2g | h |}.$$

Substituting known values,

**Equation:**

$$\begin{aligned} v &= \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m})} \\ &= 19.8 \text{ m/s.} \end{aligned}$$

**Solution for (b)**

Again  $-\Delta PE_g = \Delta KE$ . In this case there is initial kinetic energy, so

$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$ . Thus,

**Equation:**

$$mg | h | = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Rearranging gives

**Equation:**

$$\frac{1}{2}mv^2 = mg | h | + \frac{1}{2}mv_0^2.$$

This means that the final kinetic energy is the sum of the initial kinetic energy and the gravitational potential energy. Mass again cancels, and

**Equation:**

$$v = \sqrt{2g | h | + v_0^2}.$$

This equation is very similar to the kinematics equation  $v = \sqrt{v_0^2 + 2ad}$ , but it is more general—the kinematics equation is valid only for constant acceleration, whereas our equation above is valid for any path regardless of whether the object moves with a constant acceleration. Now, substituting known values gives

**Equation:**

$$\begin{aligned} v &= \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m}) + (5.00 \text{ m/s})^2} \\ &= 20.4 \text{ m/s.} \end{aligned}$$

### Discussion and Implications

First, note that mass cancels. This is quite consistent with observations made in [Falling Objects](#) that all objects fall at the same rate if friction is negligible. Second, only the speed of the roller coaster is considered; there is no information about its direction at any point. This reveals another general truth. When friction is negligible, the speed of a falling body depends only on its initial speed and height, and not on its mass or the path taken. For example, the roller coaster will have the same final speed whether it falls 20.0 m straight down or takes a more complicated path like the one in the figure. Third, and perhaps unexpectedly, the final speed in part (b) is greater than in part (a), but by far less than 5.00 m/s. Finally, note that speed can be found at *any* height along the way by simply using the appropriate value of  $h$  at the point of interest.

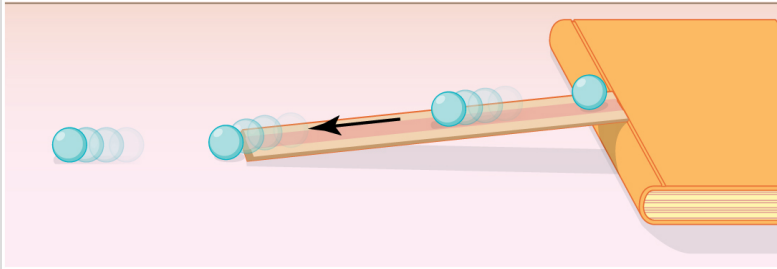
We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

### Note:

**Making Connections: Take-Home Investigation—Converting Potential to Kinetic Energy**

One can study the conversion of gravitational potential energy into kinetic energy in this experiment. On a smooth, level surface, use a ruler of the kind that has a groove running along its length and a book to make an incline (see [link](#)). Place a marble at the 10-cm position on the ruler and let it roll down the ruler. When it hits the level surface, measure the time it takes to roll one meter. Now place the marble

at the 20-cm and the 30-cm positions and again measure the times it takes to roll 1 m on the level surface. Find the velocity of the marble on the level surface for all three positions. Plot velocity squared versus the distance traveled by the marble. What is the shape of each plot? If the shape is a straight line, the plot shows that the marble's kinetic energy at the bottom is proportional to its potential energy at the release point.



A marble rolls down a ruler, and its speed on the level surface is measured.

## Section Summary

- Work done against gravity in lifting an object becomes potential energy of the object-Earth system.
- The change in gravitational potential energy,  $\Delta PE_g$ , is  $\Delta PE_g = mgh$ , with  $h$  being the increase in height and  $g$  the acceleration due to gravity.
- The gravitational potential energy of an object near Earth's surface is due to its position in the mass-Earth system. Only differences in gravitational potential energy,  $\Delta PE_g$ , have physical significance.
- As an object descends without friction, its gravitational potential energy changes into kinetic energy corresponding to increasing speed, so that  $\Delta KE = -\Delta PE_g$ .

## Conceptual Questions

**Exercise:**

**Problem:**

In [\[link\]](#), we calculated the final speed of a roller coaster that descended 20 m in height and had an initial speed of 5 m/s downhill. Suppose the roller coaster had had an initial speed of 5 m/s *uphill* instead, and it coasted uphill, stopped, and then rolled back down to a final point 20 m below the start. We would find in that case that its final speed is the same as its initial speed. Explain in terms of conservation of energy.

**Exercise:****Problem:**

Does the work you do on a book when you lift it onto a shelf depend on the path taken? On the time taken? On the height of the shelf? On the mass of the book?

**Problems & Exercises****Exercise:****Problem:**

A hydroelectric power facility (see [\[link\]](#)) converts the gravitational potential energy of water behind a dam to electric energy. (a) What is the gravitational potential energy relative to the generators of a lake of volume  $50.0 \text{ km}^3$  (mass =  $5.00 \times 10^{13} \text{ kg}$ ), given that the lake has an average height of 40.0 m above the generators? (b) Compare this with the energy stored in a 9-megaton fusion bomb.



Hydroelectric facility (credit: Denis



**Solution:**

(a)  $1.96 \times 10^{16} \text{ J}$

(b) The ratio of gravitational potential energy in the lake to the energy stored in the bomb is 0.52. That is, the energy stored in the lake is approximately half that in a 9-megaton fusion bomb.

**Exercise:**

**Problem:**

(a) How much gravitational potential energy (relative to the ground on which it is built) is stored in the Great Pyramid of Cheops, given that its mass is about  $7 \times 10^9 \text{ kg}$  and its center of mass is 36.5 m above the surrounding ground? (b) How does this energy compare with the daily food intake of a person?

**Exercise:**

**Problem:**

Suppose a 350-g kookaburra (a large kingfisher bird) picks up a 75-g snake and raises it 2.5 m from the ground to a branch. (a) How much work did the bird do on the snake? (b) How much work did it do to raise its own center of mass to the branch?

---

**Solution:**

(a) 1.8 J

(b) 8.6 J

**Exercise:**

**Problem:**

In [\[link\]](#), we found that the speed of a roller coaster that had descended 20.0 m was only slightly greater when it had an initial speed of 5.00 m/s than when it started from rest. This implies that  $\Delta PE \gg KE_i$ . Confirm this statement by taking the ratio of  $\Delta PE$  to  $KE_i$ . (Note that mass cancels.)

**Exercise:**

**Problem:**

A 100-g toy car is propelled by a compressed spring that starts it moving. The car follows the curved track in [\[link\]](#). Show that the final speed of the toy car is 0.687 m/s if its initial speed is 2.00 m/s and it coasts up the frictionless slope, gaining 0.180 m in altitude.



A toy car moves up a sloped track.  
(credit: Leszek Leszczynski, Flickr)

---

**Solution:****Equation:**

$$v_f = \sqrt{2gh + v_0^2} = \sqrt{2(9.80 \text{ m/s}^2)(-0.180 \text{ m}) + (2.00 \text{ m/s})^2} = 0.687 \text{ m/s}$$

**Exercise:****Problem:**

In a downhill ski race, surprisingly, little advantage is gained by getting a running start. (This is because the initial kinetic energy is small compared with the gain in gravitational potential energy on even small hills.) To demonstrate this, find the final speed and the time taken for a skier who skies 70.0 m along a 30° slope neglecting friction: (a) Starting from rest. (b) Starting with an initial speed of 2.50 m/s. (c) Does the answer surprise you? Discuss why it is still advantageous to get a running start in very competitive events.

**Glossary**

gravitational potential energy

the energy an object has due to its position in a gravitational field

## Conservative Forces and Potential Energy

- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces implies conservation of mechanical energy.

## Potential Energy and Conservative Forces

Work is done by a force, and some forces, such as weight, have special characteristics. A **conservative force** is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a **potential energy** (PE) for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is *conservative*. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

### **Note:**

#### Potential Energy and Conservative Forces

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

A conservative force is one for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken.

We can define a potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration

depends on the configuration, not the path followed, and is the potential energy added.

## Potential Energy of a Spring

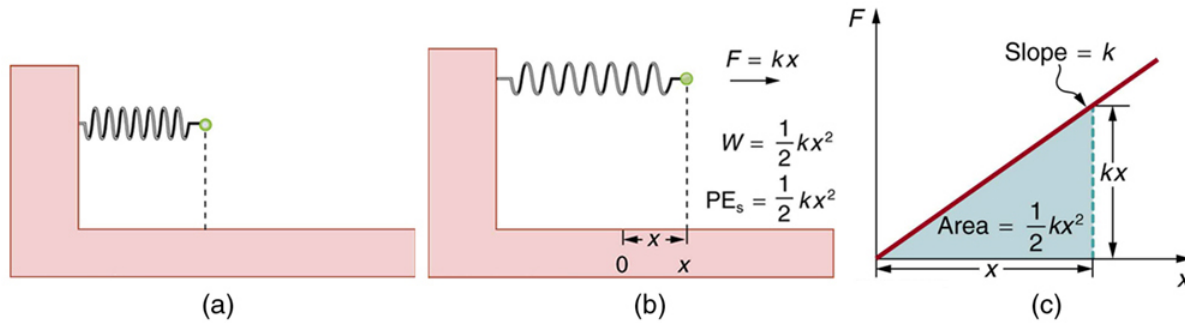
First, let us obtain an expression for the potential energy stored in a spring ( $PE_s$ ). We calculate the work done to stretch or compress a spring that obeys Hooke's law. (Hooke's law was examined in [Elasticity: Stress and Strain](#), and states that the magnitude of force  $F$  on the spring and the resulting deformation  $\Delta L$  are proportional,  $F = k\Delta L$ .) (See [\[link\]](#).) For our spring, we will replace  $\Delta L$  (the amount of deformation produced by a force  $F$ ) by the distance  $x$  that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude  $F = kx$ , where  $k$  is the spring's force constant. The force increases linearly from 0 at the start to  $kx$  in the fully stretched position. The average force is  $kx/2$ . Thus the work done in stretching or compressing the spring is

$W_s = Fd = \left(\frac{kx}{2}\right)x = \frac{1}{2}kx^2$ . Alternatively, we noted in [Kinetic Energy and the Work-Energy Theorem](#) that the area under a graph of  $F$  vs.  $x$  is the work done by the force. In [\[link\]](#)(c) we see that this area is also  $\frac{1}{2}kx^2$ . We therefore define the **potential energy of a spring**,  $PE_s$ , to be

**Equation:**

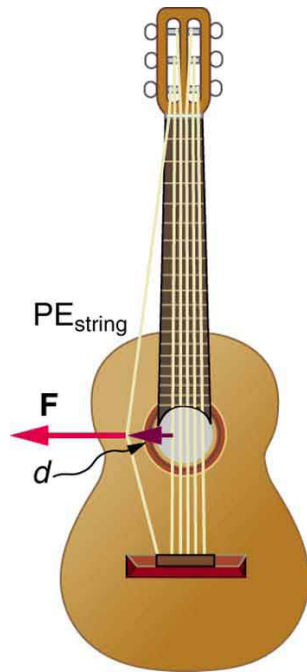
$$PE_s = \frac{1}{2}kx^2,$$

where  $k$  is the spring's force constant and  $x$  is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance  $x$ . The potential energy of the spring  $PE_s$  does not depend on the path taken; it depends only on the stretch or squeeze  $x$  in the final configuration.



- (a) An undeformed spring has no  $PE_s$  stored in it. (b) The force needed to stretch (or compress) the spring a distance  $x$  has a magnitude  $F = kx$ , and the work done to stretch (or compress) it is  $\frac{1}{2} kx^2$ . Because the force is conservative, this work is stored as potential energy ( $PE_s$ ) in the spring, and it can be fully recovered. (c) A graph of  $F$  vs.  $x$  has a slope of  $k$ , and the area under the graph is  $\frac{1}{2} kx^2$ . Thus the work done or potential energy stored is  $\frac{1}{2} kx^2$ .

The equation  $PE_s = \frac{1}{2} kx^2$  has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of **potential energy** is energy due to position, shape, or configuration. For shape or position deformations, stored energy is  $PE_s = \frac{1}{2} kx^2$ , where  $k$  is the force constant of the particular system and  $x$  is its deformation. Another example is seen in [\[link\]](#) for a guitar string.



Work is done  
to deform the  
guitar string,  
giving it  
potential  
energy.

When  
released, the  
potential  
energy is  
converted to  
kinetic  
energy and  
back to  
potential as  
the string  
oscillates  
back and  
forth. A very  
small  
fraction is  
dissipated as

sound  
energy,  
slowly  
removing  
energy from  
the string.

## Conservation of Mechanical Energy

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

**Equation:**

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta\text{KE}.$$

If only conservative forces act, then

**Equation:**

$$W_{\text{net}} = W_{\text{c}},$$

where  $W_{\text{c}}$  is the total work done by all conservative forces. Thus,

**Equation:**

$$W_{\text{c}} = \Delta\text{KE}.$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is,  $W_{\text{c}} = -\Delta\text{PE}$ . Therefore,

**Equation:**



$$-\Delta PE = \Delta KE$$

or

**Equation:**

$$\Delta KE + \Delta PE = 0.$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

**Equation:**

$$KE + PE = \text{constant}$$

or

(conservative forces only),

$$KE_i + PE_i = KE_f + PE_f$$

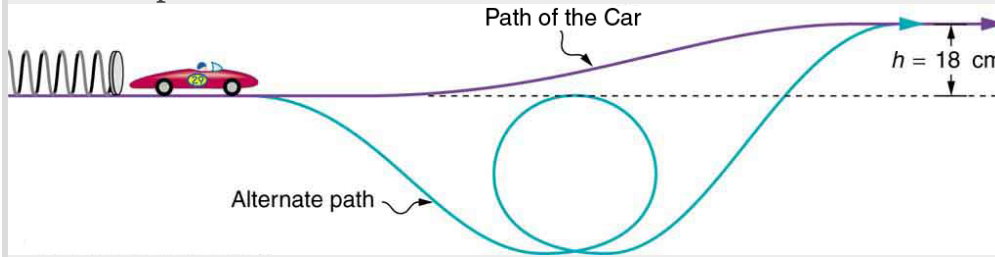
where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the **conservation of mechanical energy** principle. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its **mechanical energy**,  $(KE + PE)$ . In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE, with the total energy remaining constant.

**Example:**

**Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car**

A 0.100-kg toy car is propelled by a compressed spring, as shown in [\[link\]](#). The car follows a track that rises 0.180 m above the starting point. The spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car

is going before it starts up the slope and (b) how fast it is going at the top of the slope.



A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.

### Strategy

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

**Equation:**

$$KE_i + PE_i = KE_f + PE_f$$

or

**Equation:**

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2,$$

where  $h$  is the height (vertical position) and  $x$  is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

**Solution for (a)**

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both  $h_i$  and  $h_f$  are zero. Furthermore, the initial speed  $v_i$  is zero and the final compression of the spring  $x_f$  is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

**Equation:**

$$\frac{1}{2} kx_i^2 = \frac{1}{2} mv_f^2.$$

In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

**Equation:**

$$\begin{aligned} v_f &= \sqrt{\frac{k}{m}} x_i \\ &= \sqrt{\frac{250.0 \text{ N/m}}{0.100 \text{ kg}}} (0.0400 \text{ m}) \\ &= 2.00 \text{ m/s.} \end{aligned}$$

### **Solution for (b)**

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between. Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

**Equation:**

$$\frac{1}{2} kx_i^2 = \frac{1}{2} mv_f^2 + mgh_f.$$

This form of the equation means that the spring's initial potential energy is converted partly to gravitational potential energy and partly to kinetic energy. The final speed at the top of the slope will be less than at the bottom. Solving for  $v_f$  and substituting known values gives

**Equation:**

$$\begin{aligned}
 v_f &= \sqrt{\frac{kx_i^2}{m} - 2gh_f} \\
 &= \sqrt{\left(\frac{250.0 \text{ N/m}}{0.100 \text{ kg}}\right)(0.0400 \text{ m})^2 - 2(9.80 \text{ m/s}^2)(0.180 \text{ m})} \\
 &= 0.687 \text{ m/s.}
 \end{aligned}$$

### Discussion

Another way to solve this problem is to realize that the car's kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in [\[link\]](#). Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

### Note:

#### PhET Explorations: Energy Skate Park

Learn about conservation of energy with a skater dude! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

[https://phet.colorado.edu/sims/html/energy-skate-park-basics/latest/energy-skate-park-basics\\_en.html](https://phet.colorado.edu/sims/html/energy-skate-park-basics/latest/energy-skate-park-basics_en.html)

## Section Summary

- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy (PE) for any conservative force, just as we defined  $PE_g$  for the gravitational force.
- The potential energy of a spring is  $PE_s = \frac{1}{2}kx^2$ , where  $k$  is the spring's force constant and  $x$  is the displacement from its undeformed position.
- Mechanical energy is defined to be  $KE + PE$  for a conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,

**Equation:**

$$KE + PE = \text{constant}$$

or

$$KE_i + PE_i = KE_f + PE_f$$

where i and f denote initial and final values. This is known as the conservation of mechanical energy.

## Conceptual Questions

**Exercise:**

**Problem:** What is a conservative force?

**Exercise:**

**Problem:**

The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer dives from it, starting just before the swimmer steps on the board until just after his feet leave it.

**Exercise:**

**Problem:**

Define mechanical energy. What is the relationship of mechanical energy to nonconservative forces? What happens to mechanical energy if only conservative forces act?

**Exercise:****Problem:**

What is the relationship of potential energy to conservative force?

**Problems & Exercises****Exercise:****Problem:**

A  $5.00 \times 10^5$ -kg subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the force constant  $k$  of the spring?

---

**Solution:****Equation:**

$$7.81 \times 10^5 \text{ N/m}$$

**Exercise:****Problem:**

A pogo stick has a spring with a force constant of  $2.50 \times 10^4$  N/m, which can be compressed 12.0 cm. To what maximum height can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40.0 kg? Explicitly show how you follow the steps in the [Problem-Solving Strategies for Energy](#).

## Glossary

conservative force

a force that does the same work for any given initial and final configuration, regardless of the path followed

potential energy

energy due to position, shape, or configuration

potential energy of a spring

the stored energy of a spring as a function of its displacement; when Hooke's law applies, it is given by the expression  $\frac{1}{2}kx^2$  where  $x$  is the distance the spring is compressed or extended and  $k$  is the spring constant

conservation of mechanical energy

the rule that the sum of the kinetic energies and potential energies remains constant if only conservative forces act on and within a system

mechanical energy

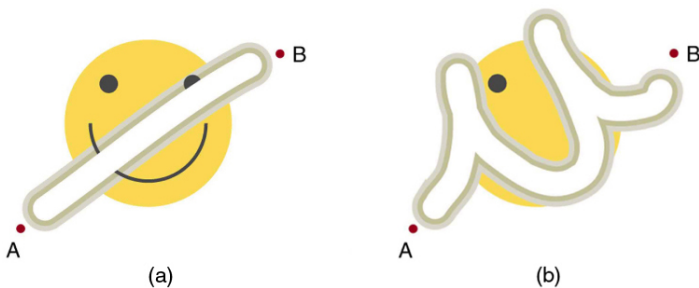
the sum of kinetic energy and potential energy

## Nonconservative Forces

- Define nonconservative forces and explain how they affect mechanical energy.
- Show how the principle of conservation of energy can be applied by treating the conservative forces in terms of their potential energies and any nonconservative forces in terms of the work they do.

### Nonconservative Forces and Friction

Forces are either conservative or nonconservative. Conservative forces were discussed in [Conservative Forces and Potential Energy](#). A **nonconservative force** is one for which work depends on the path taken. Friction is a good example of a nonconservative force. As illustrated in [\[link\]](#), work done against friction depends on the length of the path between the starting and ending points. Because of this dependence on path, there is no potential energy associated with nonconservative forces. An important characteristic is that the work done by a nonconservative force *adds or removes mechanical energy from a system*. **Friction**, for example, creates **thermal energy** that dissipates, removing energy from the system. Furthermore, even if the thermal energy is retained or captured, it cannot be fully converted back to work, so it is lost or not recoverable in that sense as well.



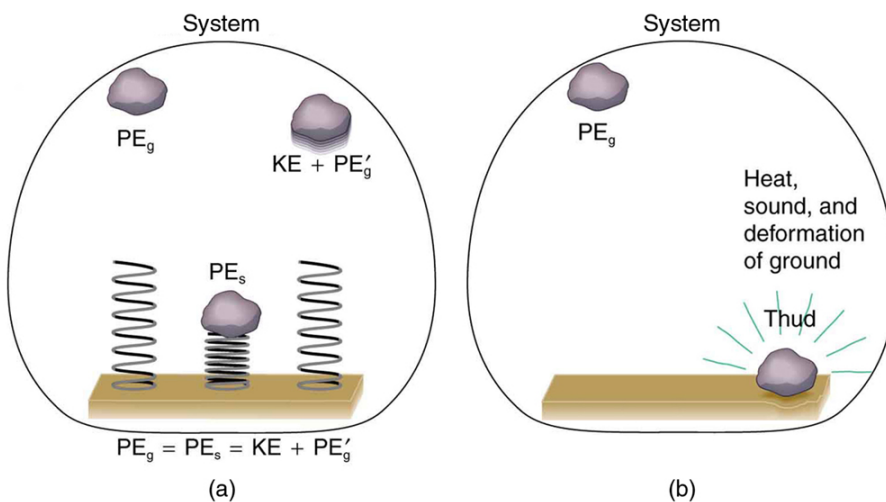
The amount of the happy face erased depends on the path taken by the eraser between points A and B, as does the work done against friction. Less work is done and less of the face



is erased for the path in (a) than for the path in (b). The force here is friction, and most of the work goes into thermal energy that subsequently leaves the system (the happy face plus the eraser). The energy expended cannot be fully recovered.

## How Nonconservative Forces Affect Mechanical Energy

*Mechanical energy may not be conserved when nonconservative forces act.* For example, when a car is brought to a stop by friction on level ground, it loses kinetic energy, which is dissipated as thermal energy, reducing its mechanical energy. [\[link\]](#) compares the effects of conservative and nonconservative forces. We often choose to understand simpler systems such as that described in [\[link\]](#)(a) first before studying more complicated systems as in [\[link\]](#)(b).



Comparison of the effects of conservative and nonconservative forces on the mechanical energy of a system. (a) A system with only conservative

forces. When a rock is dropped onto a spring, its mechanical energy remains constant (neglecting air resistance) because the force in the spring is conservative. The spring can propel the rock back to its original height, where it once again has only potential energy due to gravity. (b) A system with nonconservative forces. When the same rock is dropped onto the ground, it is stopped by nonconservative forces that dissipate its mechanical energy as thermal energy, sound, and surface distortion. The rock has lost mechanical energy.

## How the Work-Energy Theorem Applies

Now let us consider what form the work-energy theorem takes when both conservative and nonconservative forces act. We will see that the work done by nonconservative forces equals the change in the mechanical energy of a system. As noted in [Kinetic Energy and the Work-Energy Theorem](#), the work-energy theorem states that the net work on a system equals the change in its kinetic energy, or  $W_{\text{net}} = \Delta\text{KE}$ . The net work is the sum of the work by nonconservative forces plus the work by conservative forces. That is,

**Equation:**

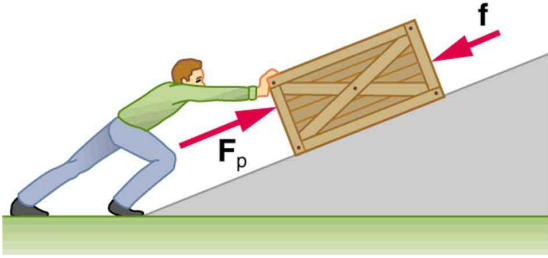
$$W_{\text{net}} = W_{\text{nc}} + W_{\text{c}},$$

so that

**Equation:**

$$W_{\text{nc}} + W_{\text{c}} = \Delta\text{KE},$$

where  $W_{\text{nc}}$  is the total work done by all nonconservative forces and  $W_{\text{c}}$  is the total work done by all conservative forces.



A person pushes a crate up a ramp, doing work on the crate. Friction and gravitational force (not shown) also do work on the crate; both forces oppose the person's push. As the crate is pushed up the ramp, it gains mechanical energy, implying that the work done by the person is greater than the work done by friction.

Consider [\[link\]](#), in which a person pushes a crate up a ramp and is opposed by friction. As in the previous section, we note that work done by a conservative force comes from a loss of gravitational potential energy, so that  $W_c = -\Delta PE$ . Substituting this equation into the previous one and solving for  $W_{nc}$  gives

**Equation:**

$$W_{nc} = \Delta KE + \Delta PE.$$

This equation means that the total mechanical energy ( $KE + PE$ ) changes by exactly the amount of work done by nonconservative forces. In [\[link\]](#), this is the work done by the person minus the work done by friction. So even if energy is not conserved for the system of interest (such as the crate), we know that an equal amount of work was done to cause the change in total mechanical energy.

We rearrange  $W_{\text{nc}} = \Delta\text{KE} + \Delta\text{PE}$  to obtain

**Equation:**

$$\text{KE}_i + \text{PE}_i + W_{\text{nc}} = \text{KE}_f + \text{PE}_f.$$

This means that the amount of work done by nonconservative forces adds to the mechanical energy of a system. If  $W_{\text{nc}}$  is positive, then mechanical energy is increased, such as when the person pushes the crate up the ramp in [\[link\]](#). If  $W_{\text{nc}}$  is negative, then mechanical energy is decreased, such as when the rock hits the ground in [\[link\]](#)(b). If  $W_{\text{nc}}$  is zero, then mechanical energy is conserved, and nonconservative forces are balanced. For example, when you push a lawn mower at constant speed on level ground, your work done is removed by the work of friction, and the mower has a constant energy.

## Applying Energy Conservation with Nonconservative Forces

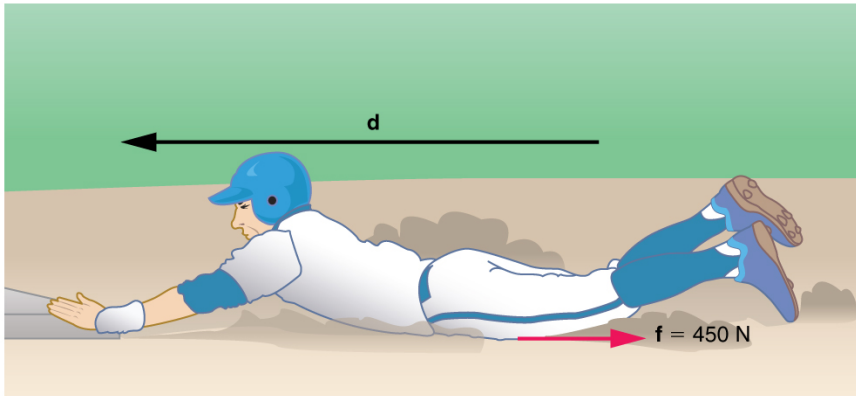
When no change in potential energy occurs, applying  $\text{KE}_i + \text{PE}_i + W_{\text{nc}} = \text{KE}_f + \text{PE}_f$  amounts to applying the work-energy theorem by setting the change in kinetic energy to be equal to the net work done on the system, which in the most general case includes both conservative and nonconservative forces. But when seeking instead to find a change in total mechanical energy in situations that involve changes in both potential and kinetic energy, the previous equation  $\text{KE}_i + \text{PE}_i + W_{\text{nc}} = \text{KE}_f + \text{PE}_f$  says that you can start by finding the change in mechanical energy that would have resulted from just the conservative forces, including the potential energy changes, and add to it the work done, with the proper sign, by any nonconservative forces involved.

### Example:

#### Calculating Distance Traveled: How Far a Baseball Player Slides

Consider the situation shown in [\[link\]](#), where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance

the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a constant 450 N.



The baseball player slides to a stop in a distance  $d$ . In the process, friction removes the player's kinetic energy by doing an amount of work  $fd$  equal to the initial kinetic energy.

### Strategy

Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction, which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because  $\mathbf{f}$  is in the opposite direction of the motion (that is,  $\theta = 180^\circ$ , and so  $\cos \theta = -1$ ). Thus  $W_{\text{nc}} = -fd$ . The equation simplifies to

### Equation:

$$\frac{1}{2}mv_i^2 - fd = 0$$

or

### Equation:

$$fd = \frac{1}{2}mv_i^2.$$

This equation can now be solved for the distance  $d$ .

### Solution

Solving the previous equation for  $d$  and substituting known values yields  
**Equation:**

$$\begin{aligned}d &= \frac{mv_i^2}{2f} \\&= \frac{(65.0 \text{ kg})(6.00 \text{ m/s})^2}{(2)(450 \text{ N})} \\&= 2.60 \text{ m.}\end{aligned}$$

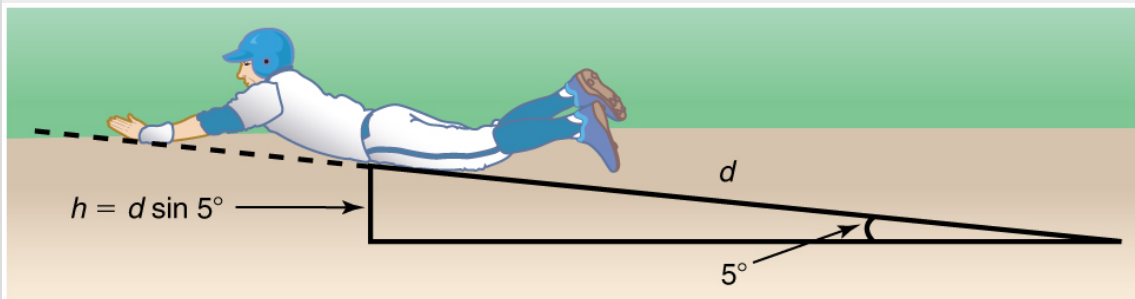
### Discussion

The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy. For example, you must work harder to stop a truck, with its large mechanical energy, than to stop a mosquito.

### Example:

#### Calculating Distance Traveled: Sliding Up an Incline

Suppose that the player from [\[link\]](#) is running up a hill having a  $5.00^\circ$  incline upward with a surface similar to that in the baseball stadium. The player slides with the same initial speed, and the frictional force is still 450 N. Determine how far he slides.



The same baseball player slides to a stop on a  $5.00^\circ$  slope.

### Strategy

In this case, the work done by the nonconservative friction force on the player reduces the mechanical energy he has from his kinetic energy at zero height, to the final mechanical energy he has by moving through

distance  $d$  to reach height  $h$  along the hill, with  $h = d \sin 5.00^\circ$ . This is expressed by the equation

**Equation:**

$$\text{KE}_i + \text{PE}_i + W_{\text{nc}} = \text{KE}_f + \text{PE}_f.$$

**Solution**

The work done by friction is again  $W_{\text{nc}} = -fd$ ; initially the potential energy is  $\text{PE}_i = mg \cdot 0 = 0$  and the kinetic energy is  $\text{KE}_i = \frac{1}{2}mv_i^2$ ; the final energy contributions are  $\text{KE}_f = 0$  for the kinetic energy and  $\text{PE}_f = mgh = mgd \sin \theta$  for the potential energy.

Substituting these values gives

**Equation:**

$$\frac{1}{2}mv_i^2 + 0 + (-fd) = 0 + mgd \sin \theta.$$

Solve this for  $d$  to obtain

**Equation:**

$$\begin{aligned} d &= \frac{(\frac{1}{2})mv_i^2}{f + mg \sin \theta} \\ &= \frac{(0.5)(65.0 \text{ kg})(6.00 \text{ m/s})^2}{450 \text{ N} + (65.0 \text{ kg})(9.80 \text{ m/s}^2) \sin (5.00^\circ)} \\ &= 2.31 \text{ m.} \end{aligned}$$

**Discussion**

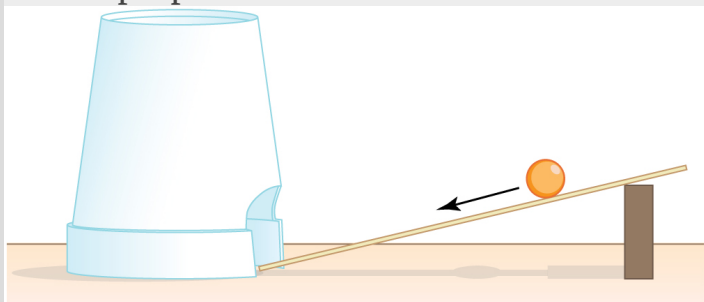
As might have been expected, the player slides a shorter distance by sliding uphill. Note that the problem could also have been solved in terms of the forces directly and the work energy theorem, instead of using the potential energy. This method would have required combining the normal force and force of gravity vectors, which no longer cancel each other because they point in different directions, and friction, to find the net force. You could then use the net force and the net work to find the distance  $d$  that reduces the kinetic energy to zero. By applying conservation of energy and using the potential energy instead, we need only consider the gravitational potential energy  $mgh$ , without combining and resolving force vectors. This simplifies the solution considerably.

**Note:****Making Connections: Take-Home Investigation—Determining Friction from the Stopping Distance**

This experiment involves the conversion of gravitational potential energy into thermal energy. Use the ruler, book, and marble from [Take-Home Investigation—Converting Potential to Kinetic Energy](#). In addition, you will need a foam cup with a small hole in the side, as shown in [\[link\]](#). From the 10-cm position on the ruler, let the marble roll into the cup positioned at the bottom of the ruler. Measure the distance  $d$  the cup moves before stopping. What forces caused it to stop? What happened to the kinetic energy of the marble at the bottom of the ruler? Next, place the marble at the 20-cm and the 30-cm positions and again measure the distance the cup moves after the marble enters it. Plot the distance the cup moves versus the initial marble position on the ruler. Is this relationship linear?

With some simple assumptions, you can use these data to find the coefficient of kinetic friction  $\mu_k$  of the cup on the table. The force of friction  $f$  on the cup is  $\mu_k N$ , where the normal force  $N$  is just the weight of the cup plus the marble. The normal force and force of gravity do no work because they are perpendicular to the displacement of the cup, which moves horizontally. The work done by friction is  $fd$ . You will need the mass of the marble as well to calculate its initial kinetic energy.

It is interesting to do the above experiment also with a steel marble (or ball bearing). Releasing it from the same positions on the ruler as you did with the glass marble, is the velocity of this steel marble the same as the velocity of the marble at the bottom of the ruler? Is the distance the cup moves proportional to the mass of the steel and glass marbles?



Rolling a marble down a ruler into a foam cup.



**Note:****PhET Explorations: The Ramp**

Explore forces, energy and work as you push household objects up and down a ramp. Lower and raise the ramp to see how the angle of inclination affects the parallel forces acting on the file cabinet. Graphs show forces, energy and work.

[The  
Ramp  
p](#)

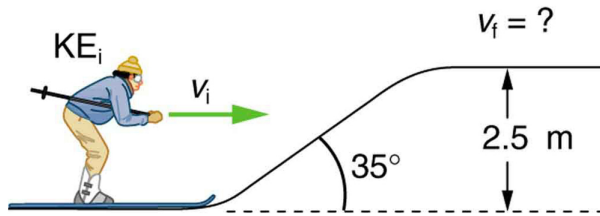
**Section Summary**

- A nonconservative force is one for which work depends on the path.
- Friction is an example of a nonconservative force that changes mechanical energy into thermal energy.
- Work  $W_{nc}$  done by a nonconservative force changes the mechanical energy of a system. In equation form,  $W_{nc} = \Delta KE + \Delta PE$  or, equivalently,  $KE_i + PE_i + W_{nc} = KE_f + PE_f$ .
- When both conservative and nonconservative forces act, energy conservation can be applied and used to calculate motion in terms of the known potential energies of the conservative forces and the work done by nonconservative forces, instead of finding the net work from the net force, or having to directly apply Newton's laws.

**Problems & Exercises****Exercise:**

**Problem:**

A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m-high rise as shown in [\[link\]](#). Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.0800. (Hint: Find the distance traveled up the incline assuming a straight-line path as shown in the figure.)



The skier's initial kinetic energy is partially used in coasting to the top of a rise.

---

**Solution:**

9.46 m/s

**Exercise:****Problem:**

(a) How high a hill can a car coast up (engine disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope  $2.5^\circ$  above the horizontal?

**Glossary**

nonconservative force

a force whose work depends on the path followed between the given initial and final configurations

friction

the force between surfaces that opposes one sliding on the other;  
friction changes mechanical energy into thermal energy

## Conservation of Energy

- Explain the law of the conservation of energy.
- Describe some of the many forms of energy.
- Define efficiency of an energy conversion process as the fraction left as useful energy or work, rather than being transformed, for example, into thermal energy.

## Law of Conservation of Energy

Energy, as we have noted, is conserved, making it one of the most important physical quantities in nature. The **law of conservation of energy** can be stated as follows:

*Total energy is constant in any process. It may change in form or be transferred from one system to another, but the total remains the same.*

We have explored some forms of energy and some ways it can be transferred from one system to another. This exploration led to the definition of two major types of energy—mechanical energy ( $KE + PE$ ) and energy transferred via work done by nonconservative forces ( $W_{nc}$ ). But energy takes *many* other forms, manifesting itself in *many* different ways, and we need to be able to deal with all of these before we can write an equation for the above general statement of the conservation of energy.

## Other Forms of Energy than Mechanical Energy

At this point, we deal with all other forms of energy by lumping them into a single group called other energy (OE). Then we can state the conservation of energy in equation form as

**Equation:**

$$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f.$$

All types of energy and work can be included in this very general statement of conservation of energy. Kinetic energy is  $KE$ , work done by a conservative force is represented by  $PE$ , work done by nonconservative forces is  $W_{nc}$ , and

all other energies are included as OE. This equation applies to all previous examples; in those situations OE was constant, and so it subtracted out and was not directly considered.

**Note:**

**Making Connections: Usefulness of the Energy Conservation Principle**

The fact that energy is conserved and has many forms makes it very important. You will find that energy is discussed in many contexts, because it is involved in all processes. It will also become apparent that many situations are best understood in terms of energy and that problems are often most easily conceptualized and solved by considering energy.

When does OE play a role? One example occurs when a person eats. Food is oxidized with the release of carbon dioxide, water, and energy. Some of this chemical energy is converted to kinetic energy when the person moves, to potential energy when the person changes altitude, and to thermal energy (another form of OE).

## **Some of the Many Forms of Energy**

What are some other forms of energy? You can probably name a number of forms of energy not yet discussed. Many of these will be covered in later chapters, but let us detail a few here. **Electrical energy** is a common form that is converted to many other forms and does work in a wide range of practical situations. Fuels, such as gasoline and food, carry **chemical energy** that can be transferred to a system through oxidation. Chemical fuel can also produce electrical energy, such as in batteries. Batteries can in turn produce light, which is a very pure form of energy. Most energy sources on Earth are in fact stored energy from the energy we receive from the Sun. We sometimes refer to this as **radiant energy**, or electromagnetic radiation, which includes visible light, infrared, and ultraviolet radiation. **Nuclear energy** comes from processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into the energy of sunlight, into electrical energy in power plants, and into the energy of the heat transfer and blast in weapons.

Atoms and molecules inside all objects are in random motion. This internal mechanical energy from the random motions is called **thermal energy**, because it is related to the temperature of the object. These and all other forms of energy can be converted into one another and can do work.

[\[link\]](#) gives the amount of energy stored, used, or released from various objects and in various phenomena. The range of energies and the variety of types and situations is impressive.

**Note:**

**Problem-Solving Strategies for Energy**

You will find the following problem-solving strategies useful whenever you deal with energy. The strategies help in organizing and reinforcing energy concepts. In fact, they are used in the examples presented in this chapter. The familiar general problem-solving strategies presented earlier—involving identifying physical principles, knowns, and unknowns, checking units, and so on—continue to be relevant here.

**Step 1.** Determine the system of interest and identify what information is given and what quantity is to be calculated. A sketch will help.

**Step 2.** Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step 3 or step 4.

**Step 3.** If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is

**Equation:**

$$KE_i + PE_i = KE_f + PE_f.$$

**Step 4.** If you know the potential energy for only some of the forces, possibly because some of them are nonconservative and do not have a potential energy, or if there are other energies that are not easily treated in terms of force and work, then the conservation of energy law in its most general form must be used.

**Equation:**

$$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f.$$

In most problems, one or more of the terms is zero, simplifying its solution. Do not calculate  $W_c$ , the work done by conservative forces; it is already incorporated in the PE terms.

**Step 5.** You have already identified the types of work and energy involved (in step 2). Before solving for the unknown, *eliminate terms wherever possible* to simplify the algebra. For example, choose  $h = 0$  at either the initial or final point, so that  $PE_g$  is zero there. Then solve for the unknown in the customary manner.

**Step 6.** *Check the answer to see if it is reasonable.* Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on. Also check to see that the numerical value obtained is reasonable. For example, the final speed of a skateboarder who coasts down a 3-m-high ramp could reasonably be 20 km/h, but *not* 80 km/h.

## Transformation of Energy

The transformation of energy from one form into others is happening all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. In a larger example, the chemical energy contained in coal is converted into thermal energy as it burns to turn water into steam in a boiler. This thermal energy in the steam in turn is converted to mechanical energy as it spins a turbine, which is connected to a generator to produce electrical energy. (In all of these examples, not all of the initial energy is converted into the forms mentioned. This important point is discussed later in this section.)

Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell (see [\[link\]](#)) produces electricity, which in turn can be used to run an electric motor. Energy is converted from the primary source of solar energy into electrical energy and then into mechanical energy.



Solar energy is converted into electrical energy by solar cells, which is used to run a motor in this solar-power aircraft. (credit: NASA)

Object/phenomenon	Energy in joules
Big Bang	$10^{68}$
Energy released in a supernova	$10^{44}$
Fusion of all the hydrogen in Earth's oceans	$10^{34}$
Annual world energy use	$4 \times 10^{20}$



Object/phenomenon	Energy in joules
Large fusion bomb (9 megaton)	$3.8 \times 10^{16}$
1 kg hydrogen (fusion to helium)	$6.4 \times 10^{14}$
1 kg uranium (nuclear fission)	$8.0 \times 10^{13}$
Hiroshima-size fission bomb (10 kiloton)	$4.2 \times 10^{13}$
90,000-ton aircraft carrier at 30 knots	$1.1 \times 10^{10}$
1 barrel crude oil	$5.9 \times 10^9$
1 ton TNT	$4.2 \times 10^9$
1 gallon of gasoline	$1.2 \times 10^8$
Daily home electricity use (developed countries)	$7 \times 10^7$
Daily adult food intake (recommended)	$1.2 \times 10^7$

Object/phenomenon	Energy in joules
1000-kg car at 90 km/h	$3.1 \times 10^5$
1 g fat (9.3 kcal)	$3.9 \times 10^4$
ATP hydrolysis reaction	$3.2 \times 10^4$
1 g carbohydrate (4.1 kcal)	$1.7 \times 10^4$
1 g protein (4.1 kcal)	$1.7 \times 10^4$
Tennis ball at 100 km/h	22
Mosquito ( $10^{-2}$ g at 0.5 m/s)	$1.3 \times 10^{-6}$
Single electron in a TV tube beam	$4.0 \times 10^{-15}$
Energy to break one DNA strand	$10^{-19}$

Energy of Various Objects and Phenomena

# Efficiency

Even though energy is conserved in an energy conversion process, the output of *useful energy* or work will be less than the energy input. The **efficiency** Eff of an energy conversion process is defined as

**Equation:**

$$\text{Efficiency(Eff)} = \frac{\text{useful energy or work output}}{\text{total energy input}} = \frac{W_{\text{out}}}{E_{\text{in}}}.$$

[\[link\]](#) lists some efficiencies of mechanical devices and human activities. In a coal-fired power plant, for example, about 40% of the chemical energy in the coal becomes useful electrical energy. The other 60% transforms into other (perhaps less useful) energy forms, such as thermal energy, which is then released to the environment through combustion gases and cooling towers.

Activity/device	Efficiency (%) <a href="#">[footnote]</a> Representative values
Cycling and climbing	20
Swimming, surface	2
Swimming, submerged	4
Shoveling	3
Weightlifting	9
Steam engine	17
Gasoline engine	30

Activity/device	Efficiency (%) <sup>[footnote]</sup> Representative values
Diesel engine	35
Nuclear power plant	35
Coal power plant	42
Electric motor	98
Compact fluorescent light	20
Gas heater (residential)	90
Solar cell	10

## Efficiency of the Human Body and Mechanical Devices

### Note:

#### PhET Explorations: Masses and Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energies for each spring.

[https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab\\_en.html](https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab_en.html)

## Section Summary

- The law of conservation of energy states that the total energy is constant in any process. Energy may change in form or be transferred from one system to another, but the total remains the same.
- When all forms of energy are considered, conservation of energy is written in equation form as

$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f$ , where OE is all **other forms of energy** besides mechanical energy.

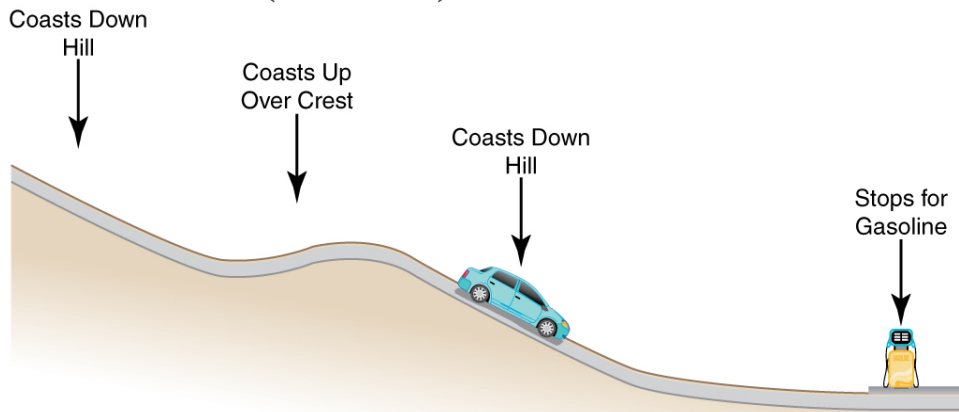
- Commonly encountered forms of energy include electric energy, chemical energy, radiant energy, nuclear energy, and thermal energy.
- Energy is often utilized to do work, but it is not possible to convert all the energy of a system to work.
- The efficiency Eff of a machine or human is defined to be  $Eff = \frac{W_{out}}{E_{in}}$ , where  $W_{out}$  is useful work output and  $E_{in}$  is the energy consumed.

## Conceptual Questions

### Exercise:

#### Problem:

Consider the following scenario. A car for which friction is *not* negligible accelerates from rest down a hill, running out of gasoline after a short distance. The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events. (See [\[link\]](#).)



A car experiencing non-negligible friction coasts down a hill, over a small crest, then downhill again, and comes to a stop at a gas station.

**Exercise:****Problem:**

Describe the energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

**Exercise:****Problem:**

Do devices with efficiencies of less than one violate the law of conservation of energy? Explain.

**Exercise:****Problem:**

List four different forms or types of energy. Give one example of a conversion from each of these forms to another form.

**Exercise:**

**Problem:** List the energy conversions that occur when riding a bicycle.

**Problems & Exercises****Exercise:****Problem:**

Using values from [\[link\]](#), how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous x rays. Later model tube TVs had shielding that absorbed x rays before they escaped and exposed viewers.)

---

**Solution:**

$4 \times 10^4$  molecules

**Exercise:****Problem:**

Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown.

---

**Solution:**

Equating  $\Delta PE_g$  and  $\Delta KE$ , we obtain

$$v = \sqrt{2gh + v_0^2} = \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m}) + (15.0 \text{ m/s})^2} = 24.8 \text{ m/s}$$

**Exercise:****Problem:**

If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from [\[link\]](#))? This is not as far-fetched as it may sound—there are thousands of nuclear bombs, and their energy can be trapped in underground explosions and converted to electricity, as natural geothermal energy is.

**Exercise:****Problem:**

(a) Use of hydrogen fusion to supply energy is a dream that may be realized in the next century. Fusion would be a relatively clean and almost limitless supply of energy, as can be seen from [\[link\]](#). To illustrate this, calculate how many years the present energy needs of the world could be supplied by one millionth of the oceans' hydrogen fusion energy. (b) How does this time compare with historically significant events, such as the duration of stable economic systems?

---

**Solution:**

(a)  $25 \times 10^6$  years

(b) This is much, much longer than human time scales.

## Glossary

law of conservation of energy

the general law that total energy is constant in any process; energy may change in form or be transferred from one system to another, but the total remains the same

electrical energy

the energy carried by a flow of charge

chemical energy

the energy in a substance stored in the bonds between atoms and molecules that can be released in a chemical reaction

radiant energy

the energy carried by electromagnetic waves

nuclear energy

energy released by changes within atomic nuclei, such as the fusion of two light nuclei or the fission of a heavy nucleus

thermal energy

the energy within an object due to the random motion of its atoms and molecules that accounts for the object's temperature

efficiency

a measure of the effectiveness of the input of energy to do work; useful energy or work divided by the total input of energy



## Power

- Calculate power by calculating changes in energy over time.
- Examine power consumption and calculations of the cost of energy consumed.

### What is Power?

*Power*—the word conjures up many images: a professional football player muscling aside his opponent, a dragster roaring away from the starting line, a volcano blowing its lava into the atmosphere, or a rocket blasting off, as in [\[link\]](#).



This powerful rocket on the Space Shuttle *Endeavor* did work and consumed energy at a very high rate. (credit: NASA)

These images of power have in common the rapid performance of work, consistent with the scientific definition of **power** ( $P$ ) as the rate at which work is done.

**Note:****Power**

Power is the rate at which work is done.

**Equation:**

$$P = \frac{W}{t}$$

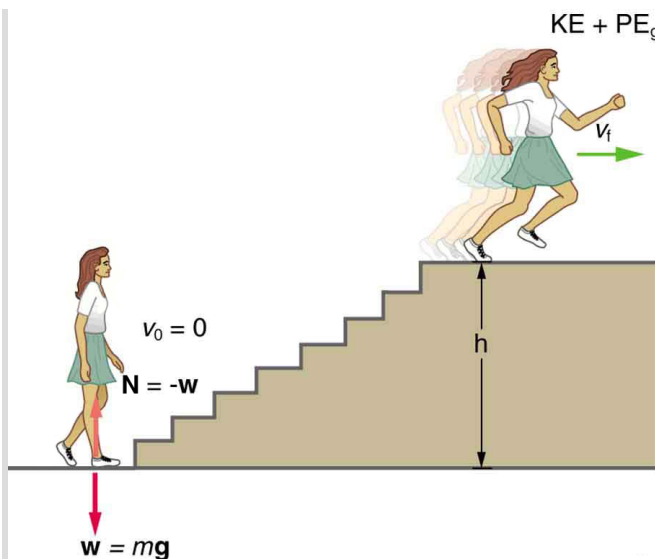
The SI unit for power is the **watt** (W), where 1 watt equals 1 joule/second ( $1 \text{ W} = 1 \text{ J/s}$ ).

Because work is energy transfer, power is also the rate at which energy is expended. A 60-W light bulb, for example, expends 60 J of energy per second. Great power means a large amount of work or energy developed in a short time. For example, when a powerful car accelerates rapidly, it does a large amount of work and consumes a large amount of fuel in a short time.

## Calculating Power from Energy

**Example:****Calculating the Power to Climb Stairs**

What is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s? (See [\[link\]](#).)



When this woman runs upstairs starting from rest, she converts the chemical energy originally from food into kinetic energy and gravitational potential energy. Her power output depends on how fast she does this.

### Strategy and Concept

The work going into mechanical energy is  $W = KE + PE$ . At the bottom of the stairs, we take both  $KE$  and  $PE_g$  as initially zero; thus,  $W = KE_f + PE_g = \frac{1}{2}mv_f^2 + mgh$ , where  $h$  is the vertical height of the stairs. Because all terms are given, we can calculate  $W$  and then divide it by time to get power.

### Solution

Substituting the expression for  $W$  into the definition of power given in the previous equation,  $P = W/t$  yields

### Equation:

$$P = \frac{W}{t} = \frac{\frac{1}{2}mv_f^2 + mgh}{t}.$$

Entering known values yields

**Equation:**

$$\begin{aligned} P &= \frac{0.5(60.0 \text{ kg})(2.00 \text{ m/s})^2 + (60.0 \text{ kg})(9.80 \text{ m/s}^2)(3.00 \text{ m})}{3.50 \text{ s}} \\ &= \frac{120 \text{ J} + 1764 \text{ J}}{3.50 \text{ s}} \\ &= 538 \text{ W}. \end{aligned}$$

**Discussion**

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.

It is impressive that this woman's useful power output is slightly less than 1 **horsepower** (1 hp = 746 W)! People can generate more than a horsepower with their leg muscles for short periods of time by rapidly converting available blood sugar and oxygen into work output. (A horse can put out 1 hp for hours on end.) Once oxygen is depleted, power output decreases and the person begins to breathe rapidly to obtain oxygen to metabolize more food—this is known as the *aerobic* stage of exercise. If the woman climbed the stairs slowly, then her power output would be much less, although the amount of work done would be the same.

**Note:****Making Connections: Take-Home Investigation—Measure Your Power Rating**

Determine your own power rating by measuring the time it takes you to climb a flight of stairs. We will ignore the gain in kinetic energy, as the above example showed that it was a small portion of the energy gain. Don't expect that your output will be more than about 0.5 hp.

**Examples of Power**

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. (See [\[link\]](#) for some examples.) Sunlight reaching Earth's surface carries a maximum power of about 1.3 kilowatts per square meter ( $\text{kW}/\text{m}^2$ ). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40% of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1000 megawatts; 1 megawatt (MW) is  $10^6$  W of electric power. But the power plant consumes chemical energy at a rate of about 2500 MW, creating heat transfer to the surroundings at a rate of 1500 MW. (See [\[link\]](#).)



Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger amount of power goes into heat transfer to the surroundings.

The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like. (credit: Kleinolive, Wikimedia Commons)

Object or Phenomenon	Power in Watts
Supernova (at peak)	$5 \times 10^{37}$
Milky Way galaxy	$10^{37}$
Crab Nebula pulsar	$10^{28}$
The Sun	$4 \times 10^{26}$

Object or Phenomenon	Power in Watts
Volcanic eruption (maximum)	$4 \times 10^{15}$
Lightning bolt	$2 \times 10^{12}$
Nuclear power plant (total electric and heat transfer)	$3 \times 10^9$
Aircraft carrier (total useful and heat transfer)	$10^8$
Dragster (total useful and heat transfer)	$2 \times 10^6$
Car (total useful and heat transfer)	$8 \times 10^4$
Football player (total useful and heat transfer)	$5 \times 10^3$
Clothes dryer	$4 \times 10^3$
Person at rest (all heat transfer)	100

Object or Phenomenon	Power in Watts
Typical incandescent light bulb (total useful and heat transfer)	60
Heart, person at rest (total useful and heat transfer)	8
Electric clock	3
Pocket calculator	$10^{-3}$

Power Output or Consumption

## Power and Energy Consumption

We usually have to pay for the energy we use. It is interesting and easy to estimate the cost of energy for an electrical appliance if its power consumption rate and time used are known. The higher the power consumption rate and the longer the appliance is used, the greater the cost of that appliance. The power consumption rate is  $P = W/t = E/t$ , where  $E$  is the energy supplied by the electricity company. So the energy consumed over a time  $t$  is

**Equation:**

$$E = Pt.$$

Electricity bills state the energy used in units of **kilowatt-hours** ( $\text{kW} \cdot \text{h}$ ), which is the product of power in kilowatts and time in hours. This unit is convenient because electrical power consumption at the kilowatt level for hours at a time is typical.



**Example:****Calculating Energy Costs**

What is the cost of running a 0.200-kW computer 6.00 h per day for 30.0 d if the cost of electricity is \$0.120 per kW · h?

**Strategy**

Cost is based on energy consumed; thus, we must find  $E$  from  $E = Pt$  and then calculate the cost. Because electrical energy is expressed in kW · h, at the start of a problem such as this it is convenient to convert the units into kW and hours.

**Solution**

The energy consumed in kW · h is

**Equation:**

$$\begin{aligned} E &= Pt = (0.200 \text{ kW})(6.00 \text{ h/d})(30.0 \text{ d}) \\ &= 36.0 \text{ kW} \cdot \text{h}, \end{aligned}$$

and the cost is simply given by

**Equation:**

$$\text{cost} = (36.0 \text{ kW} \cdot \text{h})(\$0.120 \text{ per kW} \cdot \text{h}) = \$4.32 \text{ per month.}$$

**Discussion**

The cost of using the computer in this example is neither exorbitant nor negligible. It is clear that the cost is a combination of power and time. When both are high, such as for an air conditioner in the summer, the cost is high.

The motivation to save energy has become more compelling with its ever-increasing price. Armed with the knowledge that energy consumed is the product of power and time, you can estimate costs for yourself and make the necessary value judgments about where to save energy. Either power or time must be reduced. It is most cost-effective to limit the use of high-power devices that normally operate for long periods of time, such as water heaters and air conditioners. This would not include relatively high power devices like toasters, because they are on only a few minutes per day. It would also not include electric clocks, in spite of their 24-hour-per-day

usage, because they are very low power devices. It is sometimes possible to use devices that have greater efficiencies—that is, devices that consume less power to accomplish the same task. One example is the compact fluorescent light bulb, which produces over four times more light per watt of power consumed than its incandescent cousin.

Modern civilization depends on energy, but current levels of energy consumption and production are not sustainable. The likelihood of a link between global warming and fossil fuel use (with its concomitant production of carbon dioxide), has made reduction in energy use as well as a shift to non-fossil fuels of the utmost importance. Even though energy in an isolated system is a conserved quantity, the final result of most energy transformations is waste heat transfer to the environment, which is no longer useful for doing work. As we will discuss in more detail in [Thermodynamics](#), the potential for energy to produce useful work has been “degraded” in the energy transformation.

## Section Summary

- Power is the rate at which work is done, or in equation form, for the average power  $P$  for work  $W$  done over a time  $t$ ,  $P = W/t$ .
- The SI unit for power is the watt (W), where  $1 \text{ W} = 1 \text{ J/s}$ .
- The power of many devices such as electric motors is also often expressed in horsepower (hp), where  $1 \text{ hp} = 746 \text{ W}$ .

## Conceptual Questions

### Exercise:

#### Problem:

Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

### Exercise:

**Problem:**

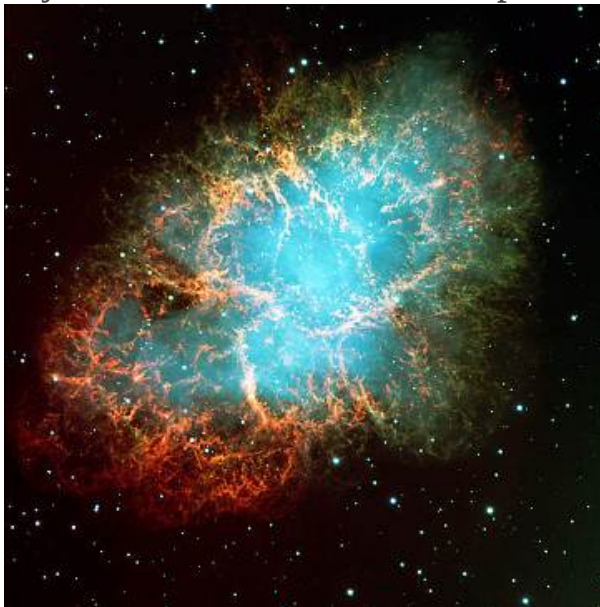
Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

**Exercise:****Problem:**

A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

**Problems & Exercises****Exercise:****Problem:**

The Crab Nebula (see [\[link\]](#)) pulsar is the remnant of a supernova that occurred in A.D. 1054. Using data from [\[link\]](#), calculate the approximate factor by which the power output of this astronomical object has declined since its explosion.



Crab Nebula (credit: ESO, via  
Wikimedia Commons)

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**Solution:**

**Equation:**

$$2 \times 10^{-10}$$

**Exercise:**

**Problem:**

Suppose a star 1000 times brighter than our Sun (that is, emitting 1000 times the power) suddenly goes supernova. Using data from [\[link\]](#): (a) By what factor does its power output increase? (b) How many times brighter than our entire Milky Way galaxy is the supernova? (c) Based on your answers, discuss whether it should be possible to observe supernovas in distant galaxies. Note that there are on the order of  $10^{11}$  observable galaxies, the average brightness of which is somewhat less than our own galaxy.

**Exercise:**

**Problem:**

A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

---

**Solution:**

(a) 40

(b) 8 million

**Exercise:**

**Problem:**

What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is \$0.0900 per  $\text{kW} \cdot \text{h}$ ?

**Exercise:**

**Problem:**

A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is \$0.110 per  $\text{kW} \cdot \text{h}$ ?

---

**Solution:**

\$149

**Exercise:**

**Problem:**

(a) What is the average power consumption in watts of an appliance that uses 5.00  $\text{kW} \cdot \text{h}$  of energy per day? (b) How many joules of energy does this appliance consume in a year?

**Exercise:**

**Problem:**

(a) What is the average useful power output of a person who does  $6.00 \times 10^6 \text{ J}$  of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

---

**Solution:**

(a) 208 W

(b) 141 s

**Exercise:**

**Problem:**

A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?

**Exercise:**

**Problem:**

(a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp = 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m-high hill in the process?

---

**Solution:**

(a) 3.20 s

(b) 4.04 s

**Exercise:**

**Problem:**

(a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is \$0.0900 per kW · h?

**Exercise:**

**Problem:**

(a) What is the available energy content, in joules, of a battery that operates a 2.00-W electric clock for 18 months? (b) How long can a battery that can supply  $8.00 \times 10^4$  J run a pocket calculator that consumes energy at the rate of  $1.00 \times 10^{-3}$  W?

---

**Solution:**

(a)  $9.46 \times 10^7$  J

(b) 2.54 y

**Exercise:****Problem:**

(a) How long would it take a  $1.50 \times 10^5$ -kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (Hint: You must find the distance the plane travels in 1200 s assuming constant acceleration.)

**Exercise:****Problem:**

Calculate the power output needed for a 950-kg car to climb a  $2.00^\circ$  slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N. Explicitly show how you follow the steps in the [Problem-Solving Strategies for Energy](#).

---

**Solution:**

Identify knowns:  $m = 950$  kg, slope angle  $\theta = 2.00^\circ$ ,  $v = 30.0$  m/s,  $f = 600$  N

Identify unknowns: power  $P$  of the car, force  $F$  that car applies to road

Solve for unknown:

$$P = \frac{W}{t} = \frac{Fd}{t} = F\left(\frac{d}{t}\right) = Fv,$$

where  $F$  is parallel to the incline and must oppose the resistive forces and the force of gravity:

$$F = f + w = 600 \text{ N} + mg \sin \theta$$

Insert this into the expression for power and solve:

$$\begin{aligned} P &= (f + mg \sin \theta)v \\ &= \left[ 600 \text{ N} + (950 \text{ kg}) \left( 9.80 \text{ m/s}^2 \right) \sin 2^\circ \right] (30.0 \text{ m/s}) \\ &= 2.77 \times 10^4 \text{ W} \end{aligned}$$

About 28 kW (or about 37 hp) is reasonable for a car to climb a gentle incline.

### **Exercise:**

#### **Problem:**

(a) Calculate the power per square meter reaching Earth's upper atmosphere from the Sun. (Take the power output of the Sun to be  $4.00 \times 10^{26} \text{ W}$ .) (b) Part of this is absorbed and reflected by the atmosphere, so that a maximum of  $1.30 \text{ kW/m}^2$  reaches Earth's surface. Calculate the area in  $\text{km}^2$  of solar energy collectors needed to replace an electric power plant that generates 750 MW if the collectors convert an average of 2.00% of the maximum power into electricity. (This small conversion efficiency is due to the devices themselves, and the fact that the sun is directly overhead only briefly.) With the same assumptions, what area would be needed to meet the United States' energy needs ( $1.05 \times 10^{20} \text{ J}$ )? Australia's energy needs ( $5.4 \times 10^{18} \text{ J}$ )? China's energy needs ( $6.3 \times 10^{19} \text{ J}$ )? (These energy consumption values are from 2006.)



## Glossary

power

the rate at which work is done

watt

(W) SI unit of power, with  $1 \text{ W} = 1 \text{ J/s}$

horsepower

an older non-SI unit of power, with  $1 \text{ hp} = 746 \text{ W}$

kilowatt-hour

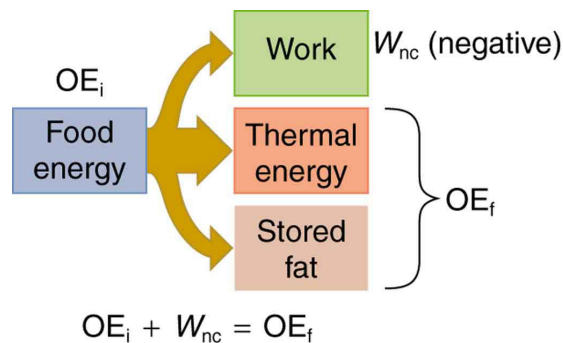
(kW · h) unit used primarily for electrical energy provided by electric utility companies

## Work, Energy, and Power in Humans

- Explain the human body's consumption of energy when at rest vs. when engaged in activities that do useful work.
- Calculate the conversion of chemical energy in food into useful work.

### Energy Conversion in Humans

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, and/or stored as chemical energy in fatty tissue. (See [\[link\]](#).) The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.



Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.

## Power Consumed at Rest

The *rate* at which the body uses food energy to sustain life and to do different activities is called the **metabolic rate**. The total energy conversion rate of a person *at rest* is called the **basal metabolic rate** (BMR) and is divided among various systems in the body, as shown in [\[link\]](#). The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.

Organ	Power consumed at rest (W)	Oxygen consumption (mL/min)	Percent of BMR
Liver & spleen	23	67	27
Brain	16	47	19
Skeletal muscle	15	45	18
Kidney	9	26	10
Heart	6	17	7
Other	16	48	19
<b>Totals</b>	85 W	250 mL/min	100%

## Basal Metabolic Rates (BMR)

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. (See [\[link\]](#).) Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food. [\[link\]](#) shows energy and oxygen consumption rates (power expended) for a variety of activities.

## Power of Doing Useful Work

Work done by a person is sometimes called **useful work**, which is *work done on the outside world*, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the mechanical energy ( $KE + PE$ ) of the system worked upon, and this is often the goal. A baseball player throwing a ball, for example, increases both the ball's kinetic and potential energy.

If a person needs more energy than they consume, such as when doing vigorous work, the body must draw upon the chemical energy stored in fat. So exercise can be helpful in losing fat. However, the amount of exercise needed to produce a loss in fat, or to burn off extra calories consumed that day, can be large, as [\[link\]](#) illustrates.

### Example:

#### Calculating Weight Loss from Exercising

If a person who normally requires an average of 12,000 kJ (3000 kcal) of food energy per day consumes 13,000 kJ per day, he will steadily gain weight. How much bicycling per day is required to work off this extra 1000 kJ?

### Solution

[\[link\]](#) states that 400 W are used when cycling at a moderate speed. The time required to work off 1000 kJ at this rate is then

**Equation:**

$$\text{Time} = \frac{\text{energy}}{\left(\frac{\text{energy}}{\text{time}}\right)} = \frac{1000 \text{ kJ}}{400 \text{ W}} = 2500 \text{ s} = 42 \text{ min.}$$

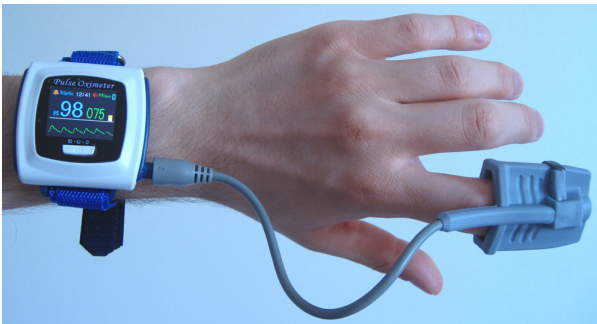
### Discussion

If this person uses more energy than he or she consumes, the person's body will obtain the needed energy by metabolizing body fat. If the person uses 13,000 kJ but consumes only 12,000 kJ, then the amount of fat loss will be

**Equation:**

$$\text{Fat loss} = (1000 \text{ kJ}) \left( \frac{1.0 \text{ g fat}}{39 \text{ kJ}} \right) = 26 \text{ g,}$$

assuming the energy content of fat to be 39 kJ/g.



A pulse oxymeter is an apparatus that measures the amount of oxygen in blood.

Oxymeters can be used to determine a person's metabolic rate, which is the rate at which food energy is converted to another form. Such

measurements can indicate the level of athletic conditioning as well as certain medical problems. (credit: UusiAjaja, Wikimedia Commons)

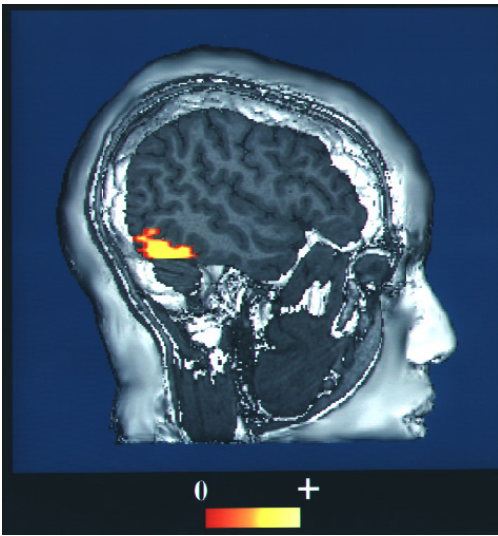
<b>Activity</b>	<b>Energy consumption in watts</b>	<b>Oxygen consumption in liters O<sub>2</sub>/min</b>
Sleeping	83	0.24
Sitting at rest	120	0.34
Standing relaxed	125	0.36
Sitting in class	210	0.60
Walking (5 km/h)	280	0.80
Cycling (13–18 km/h)	400	1.14
Shivering	425	1.21
Playing tennis	440	1.26

<b>Activity</b>	<b>Energy consumption in watts</b>	<b>Oxygen consumption in liters O<sub>2</sub>/min</b>
Swimming breaststroke	475	1.36
Ice skating (14.5 km/h)	545	1.56
Climbing stairs (116/min)	685	1.96
Cycling (21 km/h)	700	2.00
Running cross-country	740	2.12
Playing basketball	800	2.28
Cycling, professional racer	1855	5.30
Sprinting	2415	6.90

Energy and Oxygen Consumption Rates[\[footnote\]](#) (Power)  
for an average 76-kg male

All bodily functions, from thinking to lifting weights, require energy. (See [\[link\]](#).) The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract. Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and

do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all is that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.



This fMRI scan shows an increased level of energy consumption in the vision center of the brain. Here, the patient was being asked to recognize faces.  
(credit: NIH via Wikimedia Commons)

## Section Summary



- The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.
- The *rate* at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR)
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.
- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

## Conceptual Questions

### Exercise:

#### Problem:

Explain why it is easier to climb a mountain on a zigzag path rather than one straight up the side. Is your increase in gravitational potential energy the same in both cases? Is your energy consumption the same in both?

### Exercise:

#### Problem:

Do you do work on the outside world when you rub your hands together to warm them? What is the efficiency of this activity?

### Exercise:

#### Problem:

Shivering is an involuntary response to lowered body temperature. What is the efficiency of the body when shivering, and is this a desirable value?

### Exercise:

**Problem:**

Discuss the relative effectiveness of dieting and exercise in losing weight, noting that most athletic activities consume food energy at a rate of 400 to 500 W, while a single cup of yogurt can contain 1360 kJ (325 kcal). Specifically, is it likely that exercise alone will be sufficient to lose weight? You may wish to consider that regular exercise may increase the metabolic rate, whereas protracted dieting may reduce it.

**Problems & Exercises****Exercise:****Problem:**

(a) How long can you rapidly climb stairs (116/min) on the 93.0 kcal of energy in a 10.0-g pat of butter? (b) How many flights is this if each flight has 16 stairs?

---

**Solution:**

(a) 9.5 min

(b) 69 flights of stairs

**Exercise:****Problem:**

(a) What is the power output in watts and horsepower of a 70.0-kg sprinter who accelerates from rest to 10.0 m/s in 3.00 s? (b) Considering the amount of power generated, do you think a well-trained athlete could do this repetitively for long periods of time?

**Exercise:**

**Problem:**

Calculate the power output in watts and horsepower of a shot-putter who takes 1.20 s to accelerate the 7.27-kg shot from rest to 14.0 m/s, while raising it 0.800 m. (Do not include the power produced to accelerate his body.)



Shot putter at the  
Dornoch Highland  
Gathering in 2007.  
(credit: John Haslam,  
Flickr)

---

**Solution:**

641 W, 0.860 hp

**Exercise:****Problem:**

(a) What is the efficiency of an out-of-condition professor who does  $2.10 \times 10^5$  J of useful work while metabolizing 500 kcal of food energy? (b) How many food calories would a well-conditioned athlete metabolize in doing the same work with an efficiency of 20%?

**Exercise:**

**Problem:**

Energy that is not utilized for work or heat transfer is converted to the chemical energy of body fat containing about 39 kJ/g. How many grams of fat will you gain if you eat 10,000 kJ (about 2500 kcal) one day and do nothing but sit relaxed for 16.0 h and sleep for the other 8.00 h? Use data from [\[link\]](#) for the energy consumption rates of these activities.

---

**Solution:**

31 g

**Exercise:****Problem:**

Using data from [\[link\]](#), calculate the daily energy needs of a person who sleeps for 7.00 h, walks for 2.00 h, attends classes for 4.00 h, cycles for 2.00 h, sits relaxed for 3.00 h, and studies for 6.00 h. (Studying consumes energy at the same rate as sitting in class.)

**Exercise:****Problem:**

What is the efficiency of a subject on a treadmill who puts out work at the rate of 100 W while consuming oxygen at the rate of 2.00 L/min? (Hint: See [\[link\]](#).)

---

**Solution:**

14.3%

**Exercise:**

**Problem:**

Shoveling snow can be extremely taxing because the arms have such a low efficiency in this activity. Suppose a person shoveling a footpath metabolizes food at the rate of 800 W. (a) What is her useful power output? (b) How long will it take her to lift 3000 kg of snow 1.20 m? (This could be the amount of heavy snow on 20 m of footpath.) (c) How much waste heat transfer in kilojoules will she generate in the process?

**Exercise:****Problem:**

Very large forces are produced in joints when a person jumps from some height to the ground. (a) Calculate the magnitude of the force produced if an 80.0-kg person jumps from a 0.600-m-high ledge and lands stiffly, compressing joint material 1.50 cm as a result. (Be certain to include the weight of the person.) (b) In practice the knees bend almost involuntarily to help extend the distance over which you stop. Calculate the magnitude of the force produced if the stopping distance is 0.300 m. (c) Compare both forces with the weight of the person.

---

**Solution:**

(a)  $3.21 \times 10^4 \text{ N}$

(b)  $2.35 \times 10^3 \text{ N}$

(c) Ratio of net force to weight of person is 41.0 in part (a); 3.00 in part (b)

**Exercise:**

**Problem:**

Jogging on hard surfaces with insufficiently padded shoes produces large forces in the feet and legs. (a) Calculate the magnitude of the force needed to stop the downward motion of a jogger's leg, if his leg has a mass of 13.0 kg, a speed of 6.00 m/s, and stops in a distance of 1.50 cm. (Be certain to include the weight of the 75.0-kg jogger's body.) (b) Compare this force with the weight of the jogger.

**Exercise:****Problem:**

(a) Calculate the energy in kJ used by a 55.0-kg woman who does 50 deep knee bends in which her center of mass is lowered and raised 0.400 m. (She does work in both directions.) You may assume her efficiency is 20%. (b) What is the average power consumption rate in watts if she does this in 3.00 min?

---

**Solution:**

(a) 108 kJ

(b) 599 W

**Exercise:****Problem:**

Kanellos Kanellopoulos flew 119 km from Crete to Santorini, Greece, on April 23, 1988, in the *Daedalus 88*, an aircraft powered by a bicycle-type drive mechanism (see [\[link\]](#)). His useful power output for the 234-min trip was about 350 W. Using the efficiency for cycling from [\[link\]](#), calculate the food energy in kilojoules he metabolized during the flight.

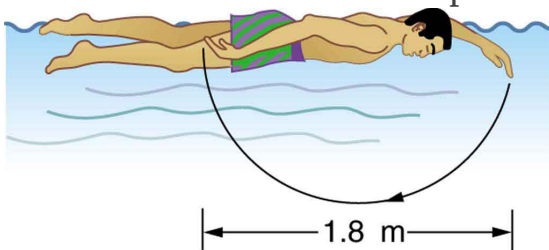


The Daedalus 88 in flight.  
(credit: NASA photo by  
Beasley)

**Exercise:**

**Problem:**

The swimmer shown in [\[link\]](#) exerts an average horizontal backward force of  $80.0\text{ N}$  with his arm during each  $1.80\text{ m}$  long stroke. (a) What is his work output in each stroke? (b) Calculate the power output of his arms if he does 120 strokes per minute.



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**Solution:**

(a)  $144\text{ J}$

(b)  $288\text{ W}$

**Exercise:**

**Problem:**

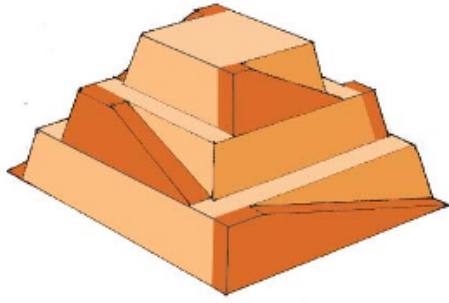
Mountain climbers carry bottled oxygen when at very high altitudes.

(a) Assuming that a mountain climber uses oxygen at twice the rate for climbing 116 stairs per minute (because of low air temperature and winds), calculate how many liters of oxygen a climber would need for 10.0 h of climbing. (These are liters at sea level.) Note that only 40% of the inhaled oxygen is utilized; the rest is exhaled. (b) How much useful work does the climber do if he and his equipment have a mass of 90.0 kg and he gains 1000 m of altitude? (c) What is his efficiency for the 10.0-h climb?

**Exercise:****Problem:**

The awe-inspiring Great Pyramid of Cheops was built more than 4500 years ago. Its square base, originally 230 m on a side, covered 13.1 acres, and it was 146 m high, with a mass of about  $7 \times 10^9$  kg. (The pyramid's dimensions are slightly different today due to quarrying and some sagging.) Historians estimate that 20,000 workers spent 20 years to construct it, working 12-hour days, 330 days per year. (a) Calculate the gravitational potential energy stored in the pyramid, given its center of mass is at one-fourth its height. (b) Only a fraction of the workers lifted blocks; most were involved in support services such as building ramps (see [\[link\]](#)), bringing food and water, and hauling blocks to the site. Calculate the efficiency of the workers who did the lifting, assuming there were 1000 of them and they consumed food energy at the rate of 300 kcal/h. What does your answer imply about how much of their work went into block-lifting, versus how much work went into friction and lifting and lowering their own bodies? (c) Calculate the mass of food that had to be supplied each day, assuming that the average worker required 3600 kcal per day and that their diet was 5% protein, 60% carbohydrate, and 35% fat. (These proportions neglect the mass of bulk and nondigestible materials consumed.)





Ancient pyramids were probably constructed using ramps as simple machines.  
(credit: Franck Monnier, Wikimedia Commons)

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**Solution:**

- (a)  $2.50 \times 10^{12}$  J
- (b) 2.52%
- (c)  $1.4 \times 10^4$  kg (14 metric tons)

**Exercise:****Problem:**

(a) How long can you play tennis on the 800 kJ (about 200 kcal) of energy in a candy bar? (b) Does this seem like a long time? Discuss why exercise is necessary but may not be sufficient to cause a person to lose weight.

**Glossary**

metabolic rate

the rate at which the body uses food energy to sustain life and to do different activities

basal metabolic rate

the total energy conversion rate of a person at rest

useful work

work done on an external system

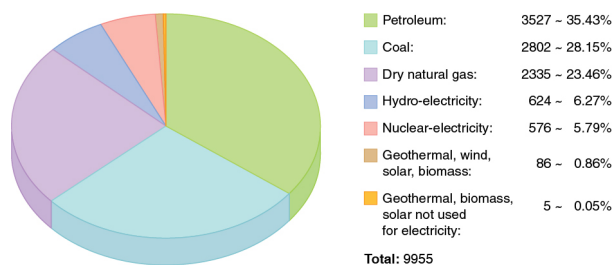
## World Energy Use

- Describe the distinction between renewable and nonrenewable energy sources.
- Explain why the inevitable conversion of energy to less useful forms makes it necessary to conserve energy resources.

Energy is an important ingredient in all phases of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. But current levels of energy consumption and production are not sustainable. About 40% of the world's energy comes from oil, and much of that goes to transportation uses. Oil prices are dependent as much upon new (or foreseen) discoveries as they are upon political events and situations around the world. The U.S., with 4.5% of the world's population, consumes 24% of the world's oil production per year; 66% of that oil is imported!

## Renewable and Nonrenewable Energy Sources

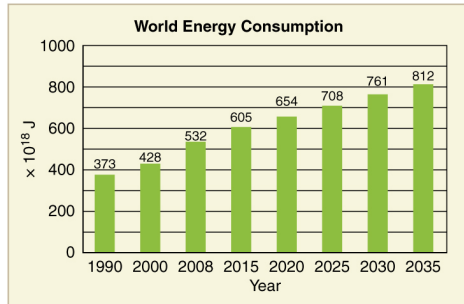
The principal energy resources used in the world are shown in [\[link\]](#). The fuel mix has changed over the years but now is dominated by oil, although natural gas and solar contributions are increasing. **Renewable forms of energy** are those sources that cannot be used up, such as water, wind, solar, and biomass. About 85% of our energy comes from nonrenewable **fossil fuels**—oil, natural gas, coal. The likelihood of a link between global warming and fossil fuel use, with its production of carbon dioxide through combustion, has made, in the eyes of many scientists, a shift to non-fossil fuels of utmost importance—but it will not be easy.



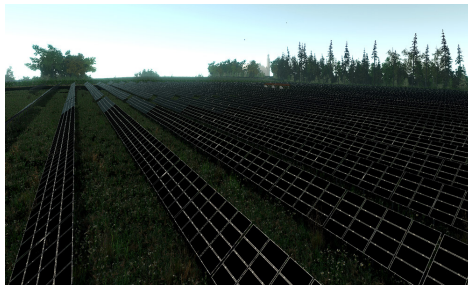
World energy consumption by source, in billions of kilowatt-hours: 2006. (credit: KVDP)

## The World's Growing Energy Needs

World energy consumption continues to rise, especially in the developing countries. (See [\[link\]](#).) Global demand for energy has tripled in the past 50 years and might triple again in the next 30 years. While much of this growth will come from the rapidly booming economies of China and India, many of the developed countries, especially those in Europe, are hoping to meet their energy needs by expanding the use of renewable sources. Although presently only a small percentage, renewable energy is growing very fast, especially wind energy. For example, Germany plans to meet 20% of its electricity and 10% of its overall energy needs with renewable resources by the year 2020. (See [\[link\]](#).) Energy is a key constraint in the rapid economic growth of China and India. In 2003, China surpassed Japan as the world's second largest consumer of oil. However, over 1/3 of this is imported. Unlike most Western countries, coal dominates the commercial energy resources of China, accounting for 2/3 of its energy consumption. In 2009 China surpassed the United States as the largest generator of CO<sub>2</sub>. In India, the main energy resources are biomass (wood and dung) and coal. Half of India's oil is imported. About 70% of India's electricity is generated by highly polluting coal. Yet there are sizeable strides being made in renewable energy. India has a rapidly growing wind energy base, and it has the largest solar cooking program in the world.



Past and projected world energy use  
(source: Based on data from U.S.  
Energy Information Administration,  
2011)



Solar cell arrays at a power plant in  
Steindorf, Germany (credit: Michael  
Betke, Flickr)

[\[link\]](#) displays the 2006 commercial energy mix by country for some of the prime energy users in the world. While non-renewable sources dominate, some countries get a sizeable percentage of their electricity from renewable resources. For example, about 67% of New Zealand's electricity demand is met by hydroelectric. Only 10% of the U.S. electricity is generated by renewable resources, primarily hydroelectric. It is difficult to determine total contributions of renewable energy in some countries with a large rural population, so these percentages in this table are left blank.

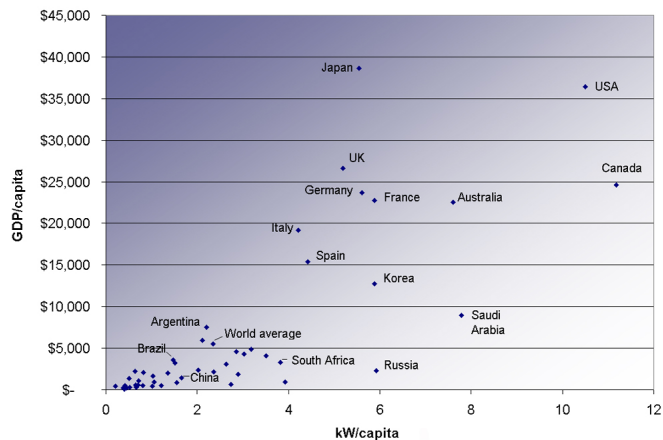
Country	Consumption, in EJ (10 <sup>18</sup> J)	Oil	Natural Gas	Coal	Nuclear	Hydro	Other Renewables
Australia	5.4	34%	17%	44%	0%	3%	1%

Country	Consumption, in EJ ( $10^{18}$ J)	Oil	Natural Gas	Coal	Nuclear	Hydro	Other Renewables
Brazil	9.6	48%	7%	5%	1%	35%	2%
China	63	22%	3%	69%	1%	6%	
Egypt	2.4	50%	41%	1%	0%	6%	
Germany	16	37%	24%	24%	11%	1%	3%
India	15	34%	7%	52%	1%	5%	
Indonesia	4.9	51%	26%	16%	0%	2%	3%
Japan	24	48%	14%	21%	12%	4%	1%
New Zealand	0.44	32%	26%	6%	0%	11%	19%
Russia	31	19%	53%	16%	5%	6%	
U.S.	105	40%	23%	22%	8%	3%	1%
<b>World</b>	<b>432</b>	<b>39%</b>	<b>23%</b>	<b>24%</b>	<b>6%</b>	<b>6%</b>	<b>2%</b>

Energy Consumption—Selected Countries (2006)

### Energy and Economic Well-being

The last two columns in this table examine the energy and electricity use per capita. Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (gross domestic product) per capita, are matched by higher levels of energy consumption per capita. This is borne out in [\[link\]](#). Increased efficiency of energy use will change this dependency. A global problem is balancing energy resource development against the harmful effects upon the environment in its extraction and use.



Power consumption per capita versus GDP per capita for various countries. Note the increase in energy usage with increasing GDP. (2007, credit: Frank van Mierlo, Wikimedia Commons)

## Conserving Energy

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As has been mentioned elsewhere, the “law of the conservation of energy” is a very useful principle in analyzing physical processes. It is a statement that cannot be proven from basic principles, but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system will always remain constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through (1) reduced activities (e.g., turning down thermostats, driving fewer kilometers) and/or (2) increasing conversion efficiencies in the performance of a particular task—such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed or created or generated, one might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat transfer to the environment and conversion to energy forms no longer useful for doing work. To state it in another way, the potential for energy to produce useful work has been “degraded” in the energy transformation. (This will be discussed in more detail in [Thermodynamics](#).)

## Section Summary

- The relative use of different fuels to provide energy has changed over the years, but fuel use is currently dominated by oil, although natural gas and solar contributions are increasing.
- Although non-renewable sources dominate, some countries meet a sizeable percentage of their electricity needs from renewable resources.
- The United States obtains only about 10% of its energy from renewable sources, mostly hydroelectric power.
- Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (Gross Domestic Product) per capita, are matched by higher levels of energy consumption per capita.
- Even though, in accordance with the law of conservation of energy, energy can never be created or destroyed, energy that can be used to do work is always partly converted to less useful forms, such as waste heat to the environment, in all of our uses of energy for practical purposes.

## Conceptual Questions

### Exercise:

#### Problem:

What is the difference between energy conservation and the law of conservation of energy? Give some examples of each.

### Exercise:

#### Problem:

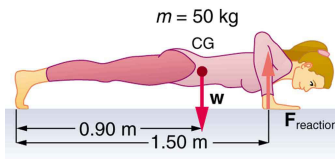
If the efficiency of a coal-fired electrical generating plant is 35%, then what do we mean when we say that energy is a conserved quantity?

## Problems & Exercises

### Exercise:

#### Problem: Integrated Concepts

(a) Calculate the force the woman in [\[link\]](#) exerts to do a push-up at constant speed, taking all data to be known to three digits. (b) How much work does she do if her center of mass rises 0.240 m? (c) What is her useful power output if she does 25 push-ups in 1 min? (Should work done lowering her body be included? See the discussion of useful work in [Work, Energy, and Power in Humans](#).



Forces involved in doing push-ups. The woman's weight acts as a force exerted downward on her center of gravity (CG).

---

### Solution:

(a) 294 N

(b) 118 J

(c) 49.0 W

### Exercise:

#### Problem: Integrated Concepts

A 75.0-kg cross-country skier is climbing a  $3.0^\circ$  slope at a constant speed of 2.00 m/s and encounters air resistance of 25.0 N. Find his power output for work done against the gravitational force and air resistance. (b) What average force does he exert backward on the snow to accomplish this? (c) If he continues to exert this force and to experience the same air resistance when he reaches a level area, how long will it take him to reach a velocity of 10.0 m/s?

**Exercise:****Problem: Integrated Concepts**

The 70.0-kg swimmer in [\[link\]](#) starts a race with an initial velocity of 1.25 m/s and exerts an average force of 80.0 N backward with his arms during each 1.80 m long stroke. (a) What is his initial acceleration if water resistance is 45.0 N? (b) What is the subsequent average resistance force from the water during the 5.00 s it takes him to reach his top velocity of 2.50 m/s? (c) Discuss whether water resistance seems to increase linearly with velocity.

---

**Solution:**

(a)  $0.500 \text{ m/s}^2$

(b) 62.5 N

(c) Assuming the acceleration of the swimmer decreases linearly with time over the 5.00 s interval, the frictional force must therefore be increasing linearly with time, since  $f = F - ma$ . If the acceleration decreases linearly with time, the velocity will contain a term dependent on time squared ( $t^2$ ). Therefore, the water resistance will not depend linearly on the velocity.

**Exercise:****Problem: Integrated Concepts**

A toy gun uses a spring with a force constant of 300 N/m to propel a 10.0-g steel ball. If the spring is compressed 7.00 cm and friction is negligible: (a) How much force is needed to compress the spring? (b) To what maximum height can the ball be shot? (c) At what angles above the horizontal may a child aim to hit a target 3.00 m away at the same height as the gun? (d) What is the gun's maximum range on level ground?

**Exercise:****Problem: Integrated Concepts**

(a) What force must be supplied by an elevator cable to produce an acceleration of  $0.800 \text{ m/s}^2$  against a 200-N frictional force, if the mass of the loaded elevator is 1500 kg? (b) How much work is done by the cable in lifting the elevator 20.0 m? (c) What is the final speed of the elevator if it starts from rest? (d) How much work went into thermal energy?

---

**Solution:**

(a)  $16.1 \times 10^3 \text{ N}$

(b)  $3.22 \times 10^5 \text{ J}$

(c) 5.66 m/s

(d) 4.00 kJ

**Exercise:****Problem: Unreasonable Results**

A car advertisement claims that its 900-kg car accelerated from rest to 30.0 m/s and drove 100 km, gaining 3.00 km in altitude, on 1.0 gal of gasoline. The average force of friction including air resistance was 700 N. Assume all values are known to three significant figures. (a) Calculate the car's efficiency. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?



**Exercise:****Problem: Unreasonable Results**

Body fat is metabolized, supplying 9.30 kcal/g, when dietary intake is less than needed to fuel metabolism. The manufacturers of an exercise bicycle claim that you can lose 0.500 kg of fat per day by vigorously exercising for 2.00 h per day on their machine. (a) How many kcal are supplied by the metabolization of 0.500 kg of fat? (b) Calculate the kcal/min that you would have to utilize to metabolize fat at the rate of 0.500 kg in 2.00 h. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

---

**Solution:**

(a)  $4.65 \times 10^3$  kcal

(b) 38.8 kcal/min

(c) This power output is higher than the highest value on [\[link\]](#), which is about 35 kcal/min (corresponding to 2415 watts) for sprinting.

(d) It would be impossible to maintain this power output for 2 hours (imagine sprinting for 2 hours!).

**Exercise:****Problem: Construct Your Own Problem**

Consider a person climbing and descending stairs. Construct a problem in which you calculate the long-term rate at which stairs can be climbed considering the mass of the person, his ability to generate power with his legs, and the height of a single stair step. Also consider why the same person can descend stairs at a faster rate for a nearly unlimited time in spite of the fact that very similar forces are exerted going down as going up. (This points to a fundamentally different process for descending versus climbing stairs.)

**Exercise:****Problem: Construct Your Own Problem**

Consider humans generating electricity by pedaling a device similar to a stationary bicycle. Construct a problem in which you determine the number of people it would take to replace a large electrical generation facility. Among the things to consider are the power output that is reasonable using the legs, rest time, and the need for electricity 24 hours per day. Discuss the practical implications of your results.

**Exercise:****Problem: Integrated Concepts**

A 105-kg basketball player crouches down 0.400 m while waiting to jump. After exerting a force on the floor through this 0.400 m, his feet leave the floor and his center of gravity rises 0.950 m above its normal standing erect position. (a) Using energy considerations, calculate his velocity when he leaves the floor. (b) What average force did he exert on the floor? (Do not neglect the force to support his weight as well as that to accelerate him.) (c) What was his power output during the acceleration phase?

---

**Solution:**

(a) 4.32 m/s

(b)  $3.47 \times 10^3$  N

(c) 8.93 kW

## **Glossary**

renewable forms of energy

those sources that cannot be used up, such as water, wind, solar, and biomass

fossil fuels

oil, natural gas, and coal

## Introduction to Linear Momentum and Collisions

class="introduction"

"Each  
rugby  
player has  
great  
momentum  
, which will  
affect the  
outcome of  
their  
collisions  
with each  
other and  
the ground.  
(credit:  
vjpaul,  
Flickr)"



We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum.

## Linear Momentum and Force

- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton's second law of motion in terms of momentum.
- Calculate momentum given mass and velocity.

### Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

**Equation:**

$$\mathbf{p} = m\mathbf{v}.$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum  $\mathbf{p}$  is a vector having the same direction as the velocity  $\mathbf{v}$ . The SI unit for momentum is  $\text{kg} \cdot \text{m/s}$ .

**Note:**

Linear Momentum

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

**Equation:**

$$\mathbf{p} = m\mathbf{v}.$$

**Example:**

### Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

#### Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum,  $p$ . (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

#### Equation:

$$p = mv$$

when only magnitudes are considered.

#### Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

#### Equation:

$$p_{\text{player}} = (110 \text{ kg})(8.00 \text{ m/s}) = 880 \text{ kg} \cdot \text{m/s}$$

#### Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

#### Equation:

$$p_{\text{ball}} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \cdot \text{m/s}$$

The ratio of the player's momentum to that of the ball is

#### Equation:

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9.$$

#### Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

## Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the “quantity of motion.” Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

**Equation:**

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

where  $\mathbf{F}_{\text{net}}$  is the net external force,  $\Delta \mathbf{p}$  is the change in momentum, and  $\Delta t$  is the change in time.

**Note:**

Newton's Second Law of Motion in Terms of Momentum

The net external force equals the change in momentum of a system divided by the time over which it changes.

**Equation:**

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}$$

**Note:****Making Connections: Force and Momentum**

Force and momentum are intimately related. Force acting over time can change momentum, and Newton's second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton's second law of motion includes the more familiar  $\mathbf{F}_{\text{net}} = m\mathbf{a}$  as a special case. We can derive this form as follows. First, note that the change in momentum  $\Delta\mathbf{p}$  is given by

**Equation:**

$$\Delta\mathbf{p} = \Delta(m\mathbf{v}).$$

If the mass of the system is constant, then

**Equation:**

$$\Delta(m\mathbf{v}) = m\Delta\mathbf{v}.$$

So that for constant mass, Newton's second law of motion becomes

**Equation:**

$$\mathbf{F}_{\text{net}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{m\Delta\mathbf{v}}{\Delta t}.$$

Because  $\frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{a}$ , we get the familiar equation

**Equation:**

$$\mathbf{F}_{\text{net}} = m\mathbf{a}$$

*when the mass of the system is constant.*



Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail; however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

**Example:**

**Calculating Force: Venus Williams' Racquet**

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

**Strategy**

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

**Equation:**

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}.$$

As noted above, when mass is constant, the change in momentum is given by

**Equation:**

$$\Delta p = m\Delta v = m(v_f - v_i).$$

In this example, the velocity just after impact and the change in time are given; thus, once  $\Delta p$  is calculated,  $F_{\text{net}} = \frac{\Delta p}{\Delta t}$  can be used to find the force.

**Solution**

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

**Equation:**

$$\begin{aligned}\Delta p &= m(v_f - v_i) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s}\end{aligned}$$

Now the magnitude of the net external force can be determined by using

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}:$$

**Equation:**

$$\begin{aligned}F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}} \\ &= 661 \text{ N} \approx 660 \text{ N},\end{aligned}$$

where we have retained only two significant figures in the final step.

### Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using  $F_{\text{net}} = ma$ , but one additional step would be required compared with the strategy used in this example.

## Section Summary

- Linear momentum (*momentum* for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum **p** is defined to be

**Equation:**

$$\mathbf{p} = m\mathbf{v},$$

where  $m$  is the mass of the system and  $\mathbf{v}$  is its velocity.

- The SI unit for momentum is  $\text{kg} \cdot \text{m/s}$ .

- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton's second law of motion is defined to be

**Equation:**

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

$\mathbf{F}_{\text{net}}$  is the net external force,  $\Delta \mathbf{p}$  is the change in momentum, and  $\Delta t$  is the change time.

## Conceptual Questions

**Exercise:**

**Problem:**

An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

**Exercise:**

**Problem:**

An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

**Exercise:**

**Problem: Professional Application**

Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.

**Exercise:**

**Problem:**

How can a small force impart the same momentum to an object as a large force?

**Problems & Exercises****Exercise:****Problem:**

(a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

---

**Solution:**

(a)  $1.50 \times 10^4 \text{ kg} \cdot \text{m/s}$

(b) 625 to 1

(c)  $6.66 \times 10^2 \text{ kg} \cdot \text{m/s}$

**Exercise:****Problem:**

(a) What is the mass of a large ship that has a momentum of  $1.60 \times 10^9 \text{ kg} \cdot \text{m/s}$ , when the ship is moving at a speed of 48.0 km/h? (b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200 m/s.

**Exercise:**

**Problem:**

(a) At what speed would a  $2.00 \times 10^4$ -kg airplane have to fly to have a momentum of  $1.60 \times 10^9 \text{ kg} \cdot \text{m/s}$  (the same as the ship's momentum in the problem above)? (b) What is the plane's momentum when it is taking off at a speed of  $60.0 \text{ m/s}$ ? (c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

---

**Solution:**

(a)  $8.00 \times 10^4 \text{ m/s}$

(b)  $1.20 \times 10^6 \text{ kg} \cdot \text{m/s}$

(c) Because the momentum of the airplane is 3 orders of magnitude smaller than of the ship, the ship will not recoil very much. The recoil would be  $-0.0100 \text{ m/s}$ , which is probably not noticeable.

**Exercise:****Problem:**

(a) What is the momentum of a garbage truck that is  $1.20 \times 10^4 \text{ kg}$  and is moving at  $10.0 \text{ m/s}$ ? (b) At what speed would an  $8.00$ -kg trash can have the same momentum as the truck?

**Exercise:****Problem:**

A runaway train car that has a mass of  $15,000 \text{ kg}$  travels at a speed of  $5.4 \text{ m/s}$  down a track. Compute the time required for a force of  $1500 \text{ N}$  to bring the car to rest.

---

**Solution:**

$54 \text{ s}$

**Exercise:****Problem:**

The mass of Earth is  $5.972 \times 10^{24}$  kg and its orbital radius is an average of  $1.496 \times 10^{11}$  m. Calculate its linear momentum.

**Glossary**

linear momentum

the product of mass and velocity

second law of motion

physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

## Impulse

- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

The effect of a force on an object depends on how long it acts, as well as how great the force is. In [\[link\]](#), a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum  $\Delta \mathbf{p}$ .

By rearranging the equation  $\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}$  to be

**Equation:**

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity  $\mathbf{F}_{\text{net}} \Delta t$  is given the name **impulse**. Impulse is the same as the change in momentum.

### **Note:**

**Impulse: Change in Momentum**

Change in momentum equals the average net external force multiplied by the time this force acts.

**Equation:**

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t$$

The quantity  $\mathbf{F}_{\text{net}} \Delta t$  is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

**Example:**

**Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall**

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of  $30^\circ$  from the perpendicular, and bounces off at an angle of  $30^\circ$  from perpendicular to the wall.

- (a) Determine the direction of the force on the wall due to each ball.
- (b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

**Strategy for (a)**

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the  $x$ -axis to be normal to the wall and to be positive in the initial direction of motion. Choose the  $y$ -axis to be



along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

**Solution for (a)**

The first ball bounces directly into the wall and exerts a force on it in the  $+x$  direction. Therefore the wall exerts a force on the ball in the  $-x$  direction. The second ball continues with the same momentum component in the  $y$  direction, but reverses its  $x$ -component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the  $-x$  direction, so the force of the wall on each ball is along the  $-x$  direction.

**Strategy for (b)**

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

**Solution for (b)**

Let  $u$  be the speed of each ball before and after collision with the wall, and  $m$  the mass of each ball. Choose the  $x$ -axis and  $y$ -axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

**Equation:**

$$p_{xi} = mu; p_{yi} = 0$$

**Equation:**

$$p_{xf} = -mu; p_{yf} = 0$$

Impulse is the change in momentum vector. Therefore the  $x$ -component of impulse is equal to  $-2mu$  and the  $y$ -component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

**Equation:**

$$p_{xi} = mu \cos 30^\circ; p_{yi} = -mu \sin 30^\circ$$

**Equation:**

$$p_{xf} = -mu \cos 30^\circ; p_{yf} = -mu \sin 30^\circ$$

It should be noted here that while  $p_x$  changes sign after the collision,  $p_y$  does not. Therefore the  $x$ -component of impulse is equal to  $-2mu \cos 30^\circ$  and the  $y$ -component of impulse is equal to zero.

The ratio of the magnitudes of the impulse imparted to the balls is

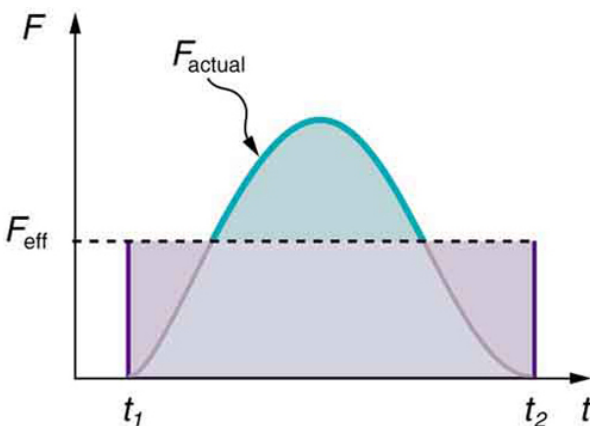
**Equation:**

$$\frac{2mu}{2mu \cos 30^\circ} = \frac{2}{\sqrt{3}} = 1.155.$$

### Discussion

The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative  $x$ -direction. Making use of Newton's third law, the force on the wall due to each ball is normal to the wall along the positive  $x$ -direction.

Our definition of impulse includes an assumption that the force is constant over the time interval  $\Delta t$ . *Forces are usually not constant.* Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force  $F_{\text{eff}}$  that produces the same result as the corresponding time-varying force. [\[link\]](#) shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times  $t_1$  and  $t_2$ . That area is equal to the area inside the rectangle bounded by  $F_{\text{eff}}$ ,  $t_1$ , and  $t_2$ . Thus the impulses and their effects are the same for both the actual and effective forces.



A graph of force versus time with time along the  $x$ -axis and force along the  $y$ -axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

**Note:**

**Making Connections: Take-Home Investigation—Hand Movement and Impulse**

Try catching a ball while “giving” with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

**Note:**

**Making Connections: Constant Force and Constant Acceleration**

The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

## Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts:

**Equation:**

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t.$$

- Forces are usually not constant over a period of time.

## Conceptual Questions

### Exercise:

#### **Problem: Professional Application**

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

### Exercise:

#### **Problem:**

While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?

### Exercise:

#### **Problem: Professional Application**

Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player's arm is not jarred as much as it would be otherwise. Explain why this is the case.

## Problems & Exercises

### Exercise:

**Problem:**

A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

---

**Solution:**

$$9.00 \times 10^3 \text{ N}$$

**Exercise:****Problem: Professional Application**

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

**Exercise:****Problem:**

A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

---

**Solution:**

a)  $2.40 \times 10^3 \text{ N}$  toward the leg

b) The force on each hand would have the same magnitude as that found in part (a) (but in opposite directions by Newton's third law) because the change in momentum and the time interval are the same.

**Exercise:**

**Problem: Professional Application**

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent's final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent's 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer's body. (d) Discuss the implications of your answers for parts (b) and (c).

**Exercise:****Problem: Professional Application**

Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

---

**Solution:**

- a) 800 kg · m/s away from the wall
- b) 1.20 m/s away from the wall

**Exercise:****Problem: Professional Application**

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of  $4.00 \times 10^3$  m/s, given the collision lasts  $6.00 \times 10^{-8}$  s.

**Exercise:****Problem: Professional Application**

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

---

**Solution:**

(a)  $1.50 \times 10^6$  N away from the dashboard

(b)  $1.00 \times 10^5$  N away from the dashboard

**Exercise:****Problem: Professional Application**

Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

**Exercise:**

**Problem:**

A cruise ship with a mass of  $1.00 \times 10^7$  kg strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

---

**Solution:**

$4.69 \times 10^5$  N in the boat's original direction of motion

**Exercise:****Problem:**

Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of  $1.76 \times 10^4$  N for  $5.50 \times 10^{-2}$  s.

**Exercise:****Problem:**

Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

---

**Solution:**

$2.10 \times 10^3$  N away from the wall

**Exercise:****Problem:**

A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?



**Exercise:****Problem:**

Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

---

**Solution:****Equation:**

$$\begin{aligned}\mathbf{p} &= m\mathbf{v} \Rightarrow p^2 = m^2v^2 \Rightarrow \frac{p^2}{m} = mv^2 \\ \Rightarrow \frac{p^2}{2m} &= \frac{1}{2}mv^2 = \text{KE} \\ KE &= \frac{p^2}{2m}\end{aligned}$$

**Exercise:****Problem:**

A ball with an initial velocity of 10 m/s moves at an angle  $60^\circ$  above the  $+x$ -direction. The ball hits a vertical wall and bounces off so that it is moving  $60^\circ$  above the  $-x$ -direction with the same speed. What is the impulse delivered by the wall?

**Exercise:****Problem:**

When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

---

**Solution:**

60.0 g

**Exercise:**

**Problem:**

A punter drops a ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle  $55^\circ$  above the horizontal. What is the impulse delivered by the foot (magnitude and direction)?

**Glossary**

change in momentum

the difference between the final and initial momentum; the mass times the change in velocity

impulse

the average net external force times the time it acts; equal to the change in momentum

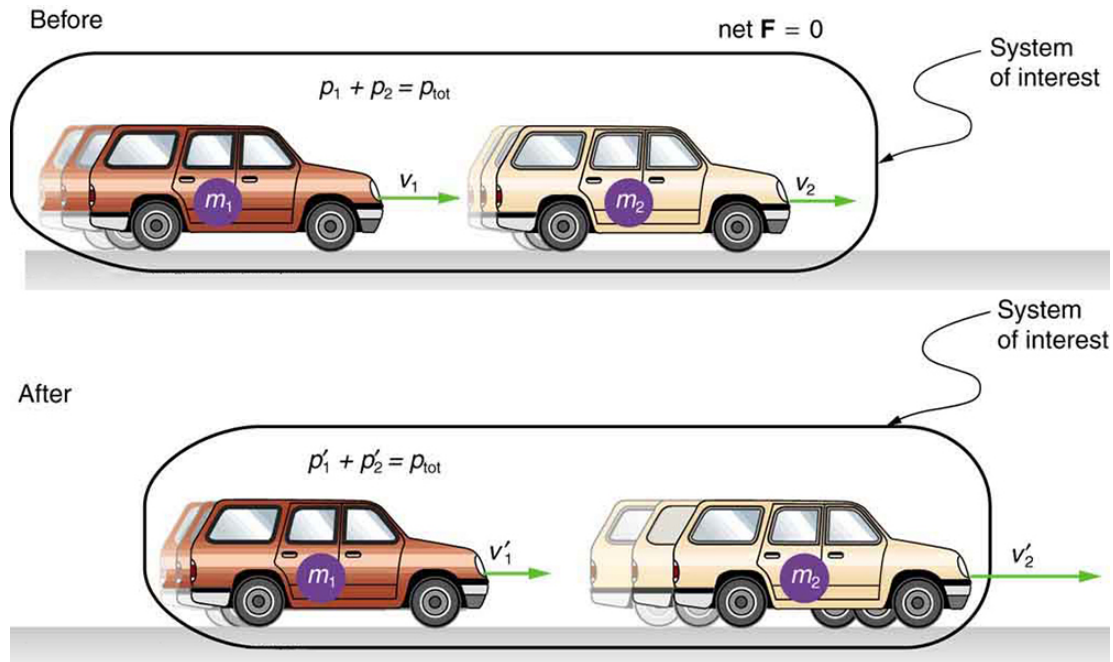
## Conservation of Momentum

- Describe the principle of conservation of momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the principle of conservation of momentum as it relates to atomic and subatomic particles.

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in [Impulse](#) and [Linear Momentum and Force](#), where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils—conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in [\[link\]](#). Both cars are coasting in the same direction when the lead car (labeled  $m_2$ ) is bumped by the trailing car (labeled  $m_1$ ). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.



A car of mass  $m_1$  moving with a velocity of  $v_1$  bumps into another car of mass  $m_2$  and velocity  $v_2$  that it is following. As a result, the first car slows down to a velocity of  $v'_1$  and the second speeds up to a velocity of  $v'_2$ . The momentum of each car is changed, but the total momentum  $p_{\text{tot}}$  of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

**Equation:**

$$\Delta p_1 = F_1 \Delta t,$$

where  $F_1$  is the force on car 1 due to car 2, and  $\Delta t$  is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling

near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

**Equation:**

$$\Delta p_2 = F_2 \Delta t,$$

where  $F_2$  is the force on car 2 due to car 1, and we assume the duration of the collision  $\Delta t$  is the same for both cars. We know from Newton's third law that  $F_2 = -F_1$ , and so

**Equation:**

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1.$$

Thus, the changes in momentum are equal and opposite, and

**Equation:**

$$\Delta p_1 + \Delta p_2 = 0.$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

**Equation:**

$$p_1 + p_2 = \text{constant},$$

**Equation:**

$$p_1 + p_2 = p'_1 + p'_2,$$

where  $p'_1$  and  $p'_2$  are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of

objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

**Equation:**

$$\mathbf{p}_{\text{tot}} = \text{constant},$$

or

**Equation:**

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}},$$

where  $\mathbf{p}_{\text{tot}}$  is the total momentum (the sum of the momenta of the individual objects in the system) and  $\mathbf{p}'_{\text{tot}}$  is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero ( $\mathbf{F}_{\text{net}} = 0$ ).

**Note:**

Conservation of Momentum Principle

**Equation:**

$$\begin{aligned}\mathbf{p}_{\text{tot}} &= \text{constant} \\ \mathbf{p}_{\text{tot}} &= \mathbf{p}'_{\text{tot}} \text{ (isolated system)}\end{aligned}$$

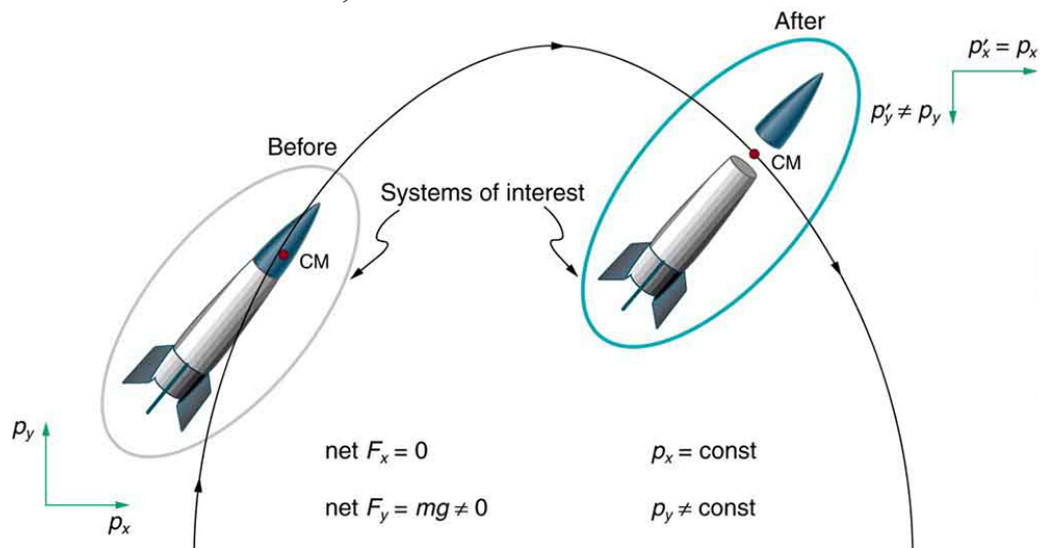
**Note:**

Isolated System

An isolated system is defined to be one for which the net external force is zero ( $\mathbf{F}_{\text{net}} = 0$ ).

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum,  $\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}_{\text{tot}}}{\Delta t}$ . For an isolated system, ( $\mathbf{F}_{\text{net}} = 0$ ); thus,  $\Delta \mathbf{p}_{\text{tot}} = 0$ , and  $\mathbf{p}_{\text{tot}}$  is constant.

We have noted that the three length dimensions in nature— $x$ ,  $y$ , and  $z$ —are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See [\[link\]](#).) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.



The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force  $F_{x-\text{net}}$  is still zero. The vertical component of the momentum is not conserved, because the net vertical force  $F_{y-\text{net}}$  is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the

space probe takes the same path it would if the separation did not occur.

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

**Note:**

**Making Connections: Take-Home Investigation—Drop of Tennis Ball and a Basketball**

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

**Note:**

**Making Connections: Take-Home Investigation—Two Tennis Balls in a Ballistic Trajectory**

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a



similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

**Note:**

**Making Connections: Conservation of Momentum and Collision**

Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

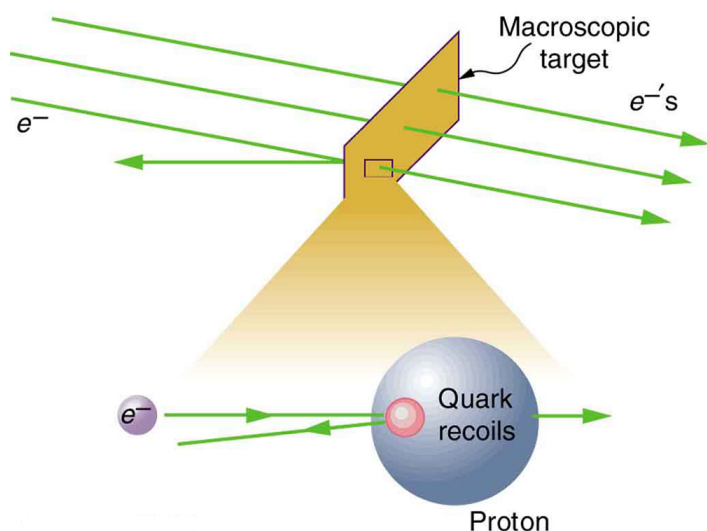
## **Subatomic Collisions and Momentum**

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results.

Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements.

Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. [\[link\]](#) below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.



A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for

quarks, electrons were observed to occasionally scatter straight backward from a proton.

## Section Summary

- The conservation of momentum principle is written  
**Equation:**

$$\mathbf{p}_{\text{tot}} = \text{constant}$$

or

**Equation:**

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}} \text{ (isolated system),}$$

$\mathbf{p}_{\text{tot}}$  is the initial total momentum and  $\mathbf{p}'_{\text{tot}}$  is the total momentum some time later.

- An isolated system is defined to be one for which the net external force is zero ( $\mathbf{F}_{\text{net}} = 0$ ).
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.
- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

## Conceptual Questions

**Exercise:**

**Problem: Professional Application**

If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.

**Exercise:**

**Problem:** Under what circumstances is momentum conserved?

**Exercise:**

**Problem:**

Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

**Exercise:**

**Problem:**

Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

**Exercise:**

**Problem: Professional Application**

Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

**Exercise:**

**Problem:**

Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

**Exercise:**

**Problem:**

Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

**Problems & Exercises****Exercise:****Problem: Professional Application**

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of  $-0.120$  m/s. (The minus indicates direction of motion.) What is their final velocity?

---

**Solution:**

0.122 m/s

**Exercise:****Problem:**

Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

**Exercise:****Problem: Professional Application**

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with

the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

---

**Solution:**

In a collision with an identical car, momentum is conserved. Afterwards  $v_f = 0$  for both cars. The change in momentum will be the same as in the crash with the tree. However, the force on the body is not determined since the time is not known. A padded stop will reduce injurious force on body.

**Exercise:**

**Problem:**

What is the velocity of a 900-kg car initially moving at 30.0 m/s, just after it hits a 150-kg deer initially running at 12.0 m/s in the same direction? Assume the deer remains on the car.

**Exercise:**

**Problem:**

A 1.80-kg falcon catches a 0.650-kg dove from behind in midair. What is their velocity after impact if the falcon's velocity is initially 28.0 m/s and the dove's velocity is 7.00 m/s in the same direction?

---

**Solution:**

22.4 m/s in the same direction as the original motion

## Glossary

conservation of momentum principle

when the net external force is zero, the total momentum of the system is conserved or constant

isolated system

a system in which the net external force is zero

quark

fundamental constituent of matter and an elementary particle

## Elastic Collisions in One Dimension

- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. [\[link\]](#) illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

### **Note:**

#### Elastic Collision

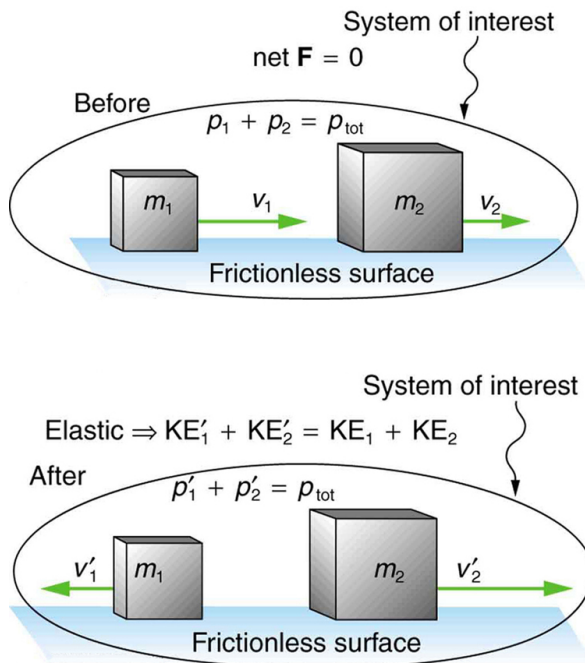
An **elastic collision** is one that conserves internal kinetic energy.

### **Note:**

#### Internal Kinetic Energy



**Internal kinetic energy** is the sum of the kinetic energies of the objects in the system.



An elastic one-dimensional  
two-object collision.  
Momentum and internal kinetic  
energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

**Equation:**

$$p_1 + p_2 = p'_1 + p'_2 \quad (F_{\text{net}} = 0)$$

or

**Equation:**

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \quad (F_{\text{net}} = 0),$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

**Equation:**

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2 \quad (\text{two-object elastic collision})$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

**Example:****Calculating Velocities Following an Elastic Collision**

Calculate the velocities of two objects following an elastic collision, given that

**Equation:**

$$m_1 = 0.500 \text{ kg}, \quad m_2 = 3.50 \text{ kg}, \quad v_1 = 4.00 \text{ m/s}, \text{ and } v_2 = 0.$$

**Strategy and Concept**

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in [\[link\]](#) where both objects are initially moving. We are asked to find two unknowns (the final velocities  $v'_1$  and  $v'_2$ ). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus  $v_2 = 0$ . Once we simplify these equations, we combine them algebraically to solve for the unknowns.

**Solution**

For this problem, note that  $v_2 = 0$  and use conservation of momentum. Thus,

**Equation:**

$$p_1 = p'_1 + p'_2$$

or

**Equation:**

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2.$$

Using conservation of internal kinetic energy and that  $v_2 = 0$ ,

**Equation:**

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2.$$

Solving the first equation (momentum equation) for  $v'_2$ , we obtain

**Equation:**

$$v'_2 = \frac{m_1}{m_2} (v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable  $v'_2$ , leaving only  $v'_1$  as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

**Equation:**

$$v'_1 = 4.00 \text{ m/s}$$

and

**Equation:**

$$v'_1 = -3.00 \text{ m/s}.$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution ( $v'_1 = -3.00 \text{ m/s}$ ) is negative, meaning that the first object bounces backward. When this negative value of  $v'_1$  is used to find the velocity of the second object after the collision, we get

**Equation:**

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00)] \text{ m/s}$$

or

**Equation:**

$$v'_2 = 1.00 \text{ m/s}.$$

### Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

### Note:

**Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision**

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

### Note:

**PhET Explorations: Collision Lab**

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum

conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

[https://phet.colorado.edu/sims/collision-lab/collision-lab\\_en.html](https://phet.colorado.edu/sims/collision-lab/collision-lab_en.html)

## Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

## Conceptual Questions

### Exercise:

**Problem:** What is an elastic collision?

## Problems & Exercises

### Exercise:

#### **Problem:**

Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

### Exercise:

#### **Problem: Professional Application**

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of  $4.00 \times 10^3$  kg, and the

second a mass of  $7.50 \times 10^3$  kg. If the two satellites collide elastically rather than dock, what is their final relative velocity?

---

**Solution:**

0.250 m/s

**Exercise:**

**Problem:**

A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

## Glossary

elastic collision

a collision that also conserves internal kinetic energy

internal kinetic energy

the sum of the kinetic energies of the objects in a system

## Inelastic Collisions in One Dimension

- Define inelastic collision.
- Explain perfectly inelastic collision.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

### Note:

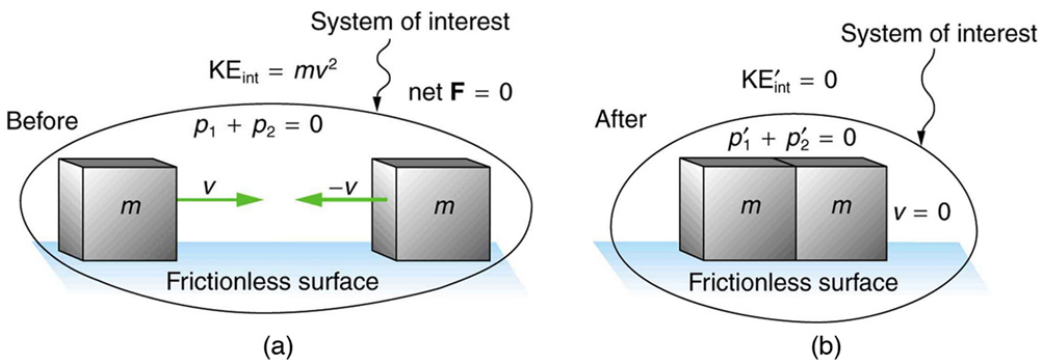
#### Inelastic Collision

An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

[\[link\]](#) shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially  $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$ . The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

**Note:****Perfectly Inelastic Collision**

A collision in which the objects stick together is sometimes called “perfectly inelastic.”



An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

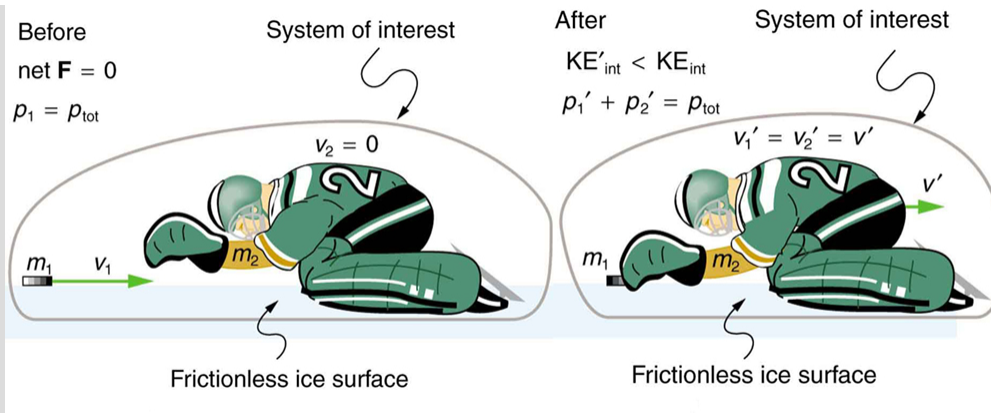
**Example:****Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie**

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision?

Assume friction between the ice and the puck-goalie system is negligible.

(See [link](#) )





An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

### Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

### Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

**Equation:**

$$p_1 + p_2 = p'_1 + p'_2$$

or

**Equation:**

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

Because the goalie is initially at rest, we know  $v_2 = 0$ . Because the goalie catches the puck, the final velocities are equal, or  $v'_1 = v'_2 = v'$ . Thus, the

conservation of momentum equation simplifies to

**Equation:**

$$m_1 v_1 = (m_1 + m_2) v_f.$$

Solving for  $v_f$  yields

**Equation:**

$$v_f = \frac{m_1}{m_1 + m_2} v_1.$$

Entering known values in this equation, we get

**Equation:**

$$v_f = \left( \frac{0.150 \text{ kg}}{0.150 \text{ kg} + 70.0 \text{ kg}} \right) (35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}.$$

**Discussion for (a)**

This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.

**Solution for (b)**

Before the collision, the internal kinetic energy  $\text{KE}_{\text{int}}$  of the system is that of the hockey puck, because the goalie is initially at rest. Therefore,  $\text{KE}_{\text{int}}$  is initially

**Equation:**

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2} m v^2 = \frac{1}{2} (0.150 \text{ kg}) (35.0 \text{ m/s})^2 \\ &= 91.9 \text{ J}. \end{aligned}$$

After the collision, the internal kinetic energy is

**Equation:**

$$\begin{aligned} \text{KE}'_{\text{int}} &= \frac{1}{2} (m + M) v^2 = \frac{1}{2} (70.15 \text{ kg}) (7.48 \times 10^{-2} \text{ m/s})^2 \\ &= 0.196 \text{ J}. \end{aligned}$$

The change in internal kinetic energy is thus

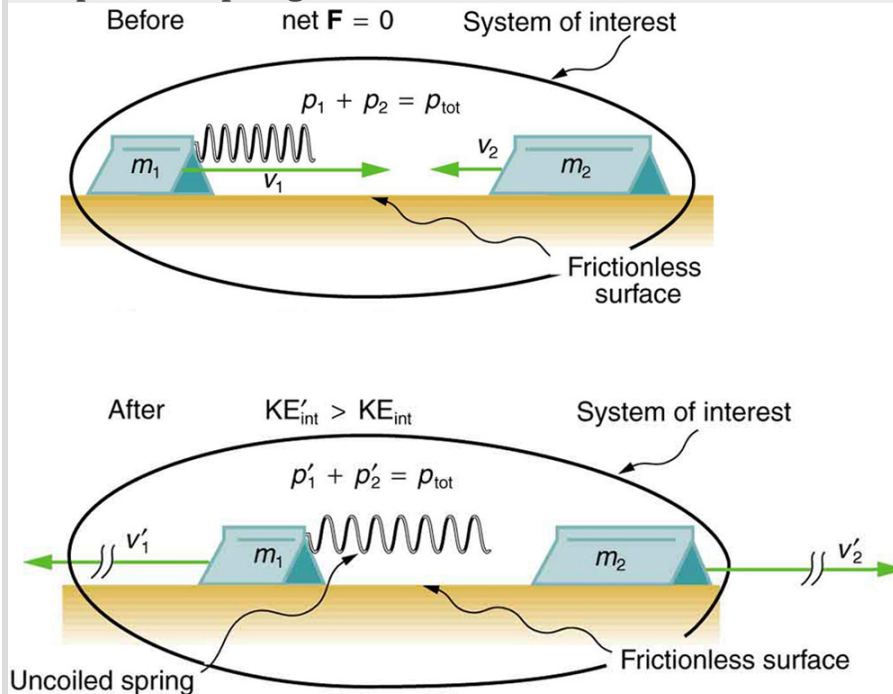
**Equation:**

$$\begin{aligned} KE_{\text{int}} - KE_{\text{int}} &= 0.196 \text{ J} - 91.9 \text{ J} \\ &= -91.7 \text{ J} \end{aligned}$$

where the minus sign indicates that the energy was lost.

### Discussion for (b)

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision.  $KE_{\text{int}}$  is mostly converted to thermal energy and sound. During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. [\[link\]](#) shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. [\[link\]](#) deals with data from such a collision.



An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in [\[link\]](#), the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the “sweet spot” on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

**Note:**

**Take-Home Experiment—Bouncing of Tennis Ball**

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend’s hand during the collision. Explain your observations and measurements.
2. The coefficient of restitution ( $c$ ) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a  $c$  of 1. For a ball bouncing off the floor (or a racquet on the floor),  $c$  can be shown to be  $c = (h/H)^{1/2}$  where  $h$  is the height to which the ball bounces and  $H$  is the height from which the ball is dropped. Determine  $c$  for the cases in Part 1 and for the case of a tennis ball bouncing off a concrete or wooden floor ( $c = 0.85$  for new tennis balls used on a tennis court).

**Example:****Calculating Final Velocity and Energy Release: Two Carts Collide**

In the collision pictured in [\[link\]](#), two carts collide inelastically. Cart 1 (denoted  $m_1$ ) carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s. Cart 2 (denoted  $m_2$  in [\[link\]](#)) has a mass of 0.500 kg and an initial velocity of  $-0.500$  m/s. After the collision, cart 1 is observed to recoil with a velocity of  $-4.00$  m/s. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

**Strategy**

We can use conservation of momentum to find the final velocity of cart 2, because  $F_{\text{net}} = 0$  (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

**Solution for (a)**

As before, the equation for conservation of momentum in a two-object system is

**Equation:**

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

The only unknown in this equation is  $v'_2$ . Solving for  $v'_2$  and substituting known values into the previous equation yields

**Equation:**

$$\begin{aligned} v'_2 &= \frac{m_1 v_1 + m_2 v_2 - m_1 v'_1}{m_2} \\ &= \frac{(0.350 \text{ kg})(2.00 \text{ m/s}) + (0.500 \text{ kg})(-0.500 \text{ m/s})}{0.500 \text{ kg}} - \frac{(0.350 \text{ kg})(-4.00 \text{ m/s})}{0.500 \text{ kg}} \\ &= 3.70 \text{ m/s.} \end{aligned}$$

**Solution for (b)**

The internal kinetic energy before the collision is

**Equation:**

$$\begin{aligned}\text{KE}_{\text{int}} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ &= \frac{1}{2}(0.350 \text{ kg})(2.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(-0.500 \text{ m/s})^2 \\ &= 0.763 \text{ J}.\end{aligned}$$

After the collision, the internal kinetic energy is

**Equation:**

$$\begin{aligned}\text{KE}'_{\text{int}} &= \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \\ &= \frac{1}{2}(0.350 \text{ kg})(-4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2 \\ &= 6.22 \text{ J}.\end{aligned}$$

The change in internal kinetic energy is thus

**Equation:**

$$\begin{aligned}\text{KE}'_{\text{int}} - \text{KE}_{\text{int}} &= 6.22 \text{ J} - 0.763 \text{ J} \\ &= 5.46 \text{ J}.\end{aligned}$$

**Discussion**

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

**Section Summary**

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.

- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

## Conceptual Questions

### Exercise:

#### Problem:

What is an inelastic collision? What is a perfectly inelastic collision?

### Exercise:

#### Problem:

Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?

### Exercise:

#### Problem:

A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

## Problems & Exercises

### Exercise:

**Problem:**

A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

---

**Solution:**

(a) 86.4 N perpendicularly away from the bumper

(b) 0.389 J

(c) 64.0%

**Exercise:****Problem:**

During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater's original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

**Exercise:****Problem: Professional Application**

Using mass and speed data from [\[link\]](#) and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

---

**Solution:**



(a) 8.06 m/s

(b) -56.0 J

(c)(i) 7.88 m/s; (ii) -223 J

**Exercise:**

**Problem:**

A battleship that is  $6.00 \times 10^7$  kg and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship's recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

**Exercise:**

**Problem: Professional Application**

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of  $4.00 \times 10^3$  kg, and the second a mass of  $7.50 \times 10^3$  kg. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

---

**Solution:**

(a) 0.163 m/s in the direction of motion of the more massive satellite

(b) 81.6 J

(c)  $8.70 \times 10^{-2}$  m/s in the direction of motion of the less massive satellite, 81.5 J. Because there are no external forces, the velocity of the center of mass of the two-satellite system is unchanged by the collision. The two velocities calculated above are the velocity of the center of mass in each of the two different individual reference frames. The loss in KE is the same in both reference frames because the KE lost to internal forces (heat, friction, etc.) is the same regardless of the coordinate system chosen.

**Exercise:**

**Problem: Professional Application**

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

**Exercise:**

**Problem: Professional Application**

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

---

**Solution:**

0.704 m/s

−2.25 m/s

**Exercise:**

**Problem:**

A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle's kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

---

**Solution:**

(a) 4.58 m/s away from the bullet

(b) 31.5 J

(c)  $-0.491$  m/s

(d) 3.38 J

**Exercise:****Problem: Professional Application**

One of the waste products of a nuclear reactor is plutonium-239 ( $^{239}\text{Pu}$ ). This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus ( $^4\text{He} + ^{235}\text{U}$ ), the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is  $8.40 \times 10^{-13}$  J and is entirely converted to kinetic energy of the helium and uranium nuclei.

The mass of the helium nucleus is  $6.68 \times 10^{-27}$  kg, while that of the uranium is  $3.92 \times 10^{-25}$  kg (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

**Exercise:**

**Problem: Professional Application**

The Moon's craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of  $5.00 \times 10^{12}$  kg (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is  $7.36 \times 10^{22}$  kg)? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

---

**Solution:**

(a)  $1.02 \times 10^{-6}$  m/s

(b)  $5.63 \times 10^{20}$  J (almost all KE lost)

(c) Recoil speed is  $6.79 \times 10^{-17}$  m/s, energy lost is  $6.25 \times 10^9$  J. The plume will not affect the momentum result because the plume is still part of the Moon system. The plume may affect the kinetic energy result because a significant part of the initial kinetic energy may be transferred to the kinetic energy of the plume particles.

**Exercise:**

**Problem: Professional Application**

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of  $-3.50$  m/s. What is their velocity just after impact if they cling together?

**Exercise:****Problem:**

What is the speed of a garbage truck that is  $1.20 \times 10^4$  kg and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

---

**Solution:**

24.8 m/s

**Exercise:****Problem:**

During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

**Exercise:****Problem:**

(a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown's ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

---

**Solution:**

(a) 4.00 kg

(b) 210 J

(c) The clown does work to throw the barbell, so the kinetic energy comes from the muscles of the clown. The muscles convert the chemical potential energy of ATP into kinetic energy.

**Glossary**

inelastic collision

a collision in which internal kinetic energy is not conserved

perfectly inelastic collision

a collision in which the colliding objects stick together

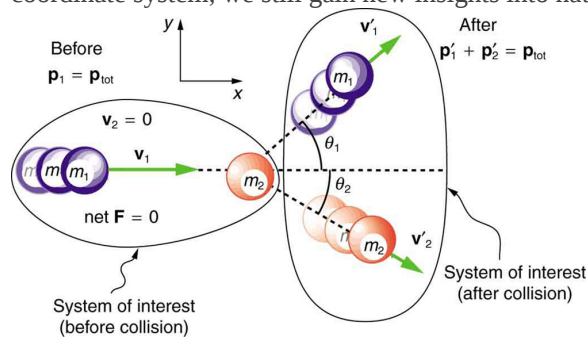
## Collisions of Point Masses in Two Dimensions

- Discuss two dimensional collisions as an extension of one dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along x-axis and y-axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity, and scattering angle.

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that  $\mathbf{F}_{\text{net}} = 0$ , so that momentum  $\mathbf{p}$  is conserved. The simplest collision is one in which one of the particles is initially at rest. (See [\[link\]](#).) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in [\[link\]](#). Because momentum is conserved, the components of momentum along the  $x$ - and  $y$ -axes ( $p_x$  and  $p_y$ ) will also be conserved, but with the chosen coordinate system,  $p_y$  is initially zero and  $p_x$  is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)



A two-dimensional collision with the coordinate system chosen so that  $m_2$  is initially at rest and  $v_1$  is parallel to the  $x$ -axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the  $x$ -axis, the equation for conservation of momentum is

**Equation:**

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}.$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

**Equation:**

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

But because particle 2 is initially at rest, this equation becomes

**Equation:**

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

The components of the velocities along the  $x$ -axis have the form  $v \cos \theta$ . Because particle 1 initially moves along the  $x$ -axis, we find  $v_{1x} = v_1$ .

Conservation of momentum along the  $x$ -axis gives the following equation:

**Equation:**

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2,$$

where  $\theta_1$  and  $\theta_2$  are as shown in [\[link\]](#).

**Note:**

Conservation of Momentum along the  $x$ -axis

**Equation:**

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$$

Along the  $y$ -axis, the equation for conservation of momentum is

**Equation:**

$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y}$$

or

**Equation:**

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}.$$

But  $v_{1y}$  is zero, because particle 1 initially moves along the  $x$ -axis. Because particle 2 is initially at rest,  $v_{2y}$  is also zero. The equation for conservation of momentum along the  $y$ -axis becomes

**Equation:**

$$0 = m_1 v'_{1y} + m_2 v'_{2y}.$$



The components of the velocities along the  $y$ -axis have the form  $v \sin \theta$ .

Thus, conservation of momentum along the  $y$ -axis gives the following equation:

**Equation:**

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2.$$

**Note:**

Conservation of Momentum along the  $y$ -axis

**Equation:**

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$$

The equations of conservation of momentum along the  $x$ -axis and  $y$ -axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

**Example:**

**Determining the Final Velocity of an Unseen Object from the Scattering of Another Object**

Suppose the following experiment is performed. A 0.250-kg object ( $m_1$ ) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg ( $m_2$ ). The 0.250-kg object emerges from the room at an angle of  $45.0^\circ$  with its incoming direction.

The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity ( $v'_2$  and  $\theta_2$ ) of the 0.400-kg object after the collision.

**Strategy**

Momentum is conserved because the surface is frictionless. The coordinate system shown in [\[link\]](#) is one in which  $m_2$  is originally at rest and the initial velocity is parallel to the  $x$ -axis, so that conservation of momentum along the  $x$ - and  $y$ -axes is applicable.

Everything is known in these equations except  $v'_2$  and  $\theta_2$ , which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the  $x$ - and  $y$ -directions.

**Solution**

Solving  $m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$  for  $v'_2 \cos \theta_2$  and  $0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$  for  $v'_2 \sin \theta_2$  and taking the ratio yields an equation (in which  $\theta_2$  is the only unknown quantity. Applying the identity ( $\tan \theta = \frac{\sin \theta}{\cos \theta}$ ), we obtain:

**Equation:**

$$\tan \theta_2 = \frac{v'_1 \sin \theta_1}{v'_1 \cos \theta_1 - v_1}.$$

Entering known values into the previous equation gives

**Equation:**

$$\tan \theta_2 = \frac{(1.50 \text{ m/s})(0.7071)}{(1.50 \text{ m/s})(0.7071) - 2.00 \text{ m/s}} = -1.129.$$

Thus,

**Equation:**

$$\theta_2 = \tan^{-1}(-1.129) = 311.5^\circ \approx 312^\circ.$$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that  $m_2$  is scattered to the right in [\[link\]](#), as expected (this angle is in the fourth quadrant). Either equation for the  $x$ - or  $y$ -axis can now be used to solve for  $v'_2$ , but the latter equation is easiest because it has fewer terms.

**Equation:**

$$v'_2 = -\frac{m_1}{m_2} v_1 \frac{\sin \theta_1}{\sin \theta_2}$$

Entering known values into this equation gives

**Equation:**

$$v'_2 = -\left(\frac{0.250 \text{ kg}}{0.400 \text{ kg}}\right)(1.50 \text{ m/s})\left(\frac{0.7071}{-0.7485}\right).$$

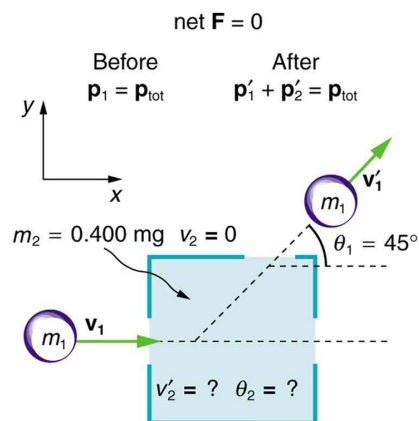
Thus,

**Equation:**

$$v'_2 = 0.886 \text{ m/s}.$$

**Discussion**

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.



A collision taking place in a dark room is explored in [\[link\]](#).

The incoming object  $m_1$  is scattered by an initially stationary object. Only the stationary object's mass  $m_2$  is known. By measuring the angle and speed at which  $m_1$  emerges from the room, it is possible to calculate the magnitude and

direction of the initially  
stationary object's velocity after  
the collision.

## Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to [\[link\]](#) for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 ( $m_2$ ) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

**Equation:**

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2.$$

Because the masses are equal,  $m_1 = m_2 = m$ . Algebraic manipulation (left to the reader) of conservation of momentum in the  $x$ - and  $y$ -directions can show that

**Equation:**

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv_1v_2' \cos(\theta_1 - \theta_2).$$

(Remember that  $\theta_2$  is negative here.) The two preceding equations can both be true only if

**Equation:**

$$mv_1v_2' \cos(\theta_1 - \theta_2) = 0.$$

There are three ways that this term can be zero. They are

- $v_1 = 0$ : head-on collision; incoming ball stops
- $v_2' = 0$ : no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$ : angle of separation ( $\theta_1 - \theta_2$ ) is  $90^\circ$  after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to  $90^\circ$  after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

### Note:

#### Connections to Nuclear and Particle Physics

Two-dimensional collision experiments have revealed much of what we know about subatomic particles, as we shall see in [Medical Applications of Nuclear Physics](#) and [Particle Physics](#). Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

## Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the  $x$ -axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the  $x$ -axis), stated by  $m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$  and along the direction perpendicular to the initial direction (the  $y$ -axis) stated by  $0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$ .
- The internal kinetic before and after the collision of two objects that have equal masses is

**Equation:**

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v'^2_1 + \frac{1}{2} m v'^2_2 + m v_1 v'_2 \cos(\theta_1 - \theta_2).$$

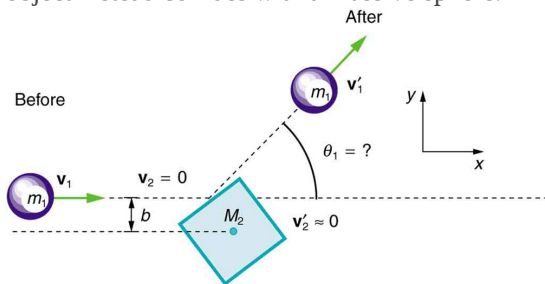
- Point masses are structureless particles that cannot spin.

## Conceptual Questions

**Exercise:**

**Problem:**

[\[link\]](#) shows a cube at rest and a small object heading toward it. (a) Describe the directions (angle  $\theta_1$ ) at which the small object can emerge after colliding elastically with the cube. How does  $\theta_1$  depend on  $b$ , the so-called impact parameter? Ignore any effects that might be due to rotation after the collision, and assume that the cube is much more massive than the small object. (b) Answer the same questions if the small object instead collides with a massive sphere.



A small object approaches a collision with a much more massive cube, after which its velocity has the direction  $\theta_1$ . The angles at which the small object can be scattered are determined by the shape of the object it strikes and the impact parameter  $b$ .

## Problems & Exercises

**Exercise:**

**Problem:**

Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of  $30.0^\circ$ , what is the velocity (magnitude and direction) of the second puck? (You may use the result that  $\theta_1 - \theta_2 = 90^\circ$  for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

**Solution:**

(a) 3.00 m/s,  $60^\circ$  below  $x$ -axis

(b) Find speed of first puck after collision:

$$0 = mv_1' \sin 30^\circ - mv_2' \sin 60^\circ \Rightarrow v_1' = v_2' \frac{\sin 60^\circ}{\sin 30^\circ} = 5.196 \text{ m/s}$$

Verify that ratio of initial to final KE equals one: 
$$\left. \begin{aligned} \text{KE} &= \frac{1}{2}mv_1^2 = 18m \text{ J} \\ \text{KE} &= \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = 18m \text{ J} \end{aligned} \right\} \frac{\text{KE}}{\text{KE}} = 1.00$$

**Exercise:****Problem:**

Confirm that the results of the example [\[link\]](#) do conserve momentum in both the  $x$ - and  $y$ -directions.

**Exercise:****Problem:**

A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of  $20.0^\circ$  above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

**Solution:**

(a)  $-2.26 \text{ m/s}$

(b)  $7.63 \times 10^3 \text{ J}$

(c) The ground will exert a normal force to oppose recoil of the cannon in the vertical direction. The momentum in the vertical direction is transferred to the earth. The energy is transferred into the ground, making a dent where the cannon is. After long barrages, cannon have erratic aim because the ground is full of divots.

**Exercise:****Problem: Professional Application**

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of  $85.0^\circ$  to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.

**Exercise:****Problem: Professional Application**

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei ( ${}^4\text{He}$ ) from gold-197 nuclei ( ${}^{197}\text{Au}$ ). The energy of the incoming helium nucleus was  $8.00 \times 10^{-13} \text{ J}$ , and the masses of the helium and gold nuclei were  $6.68 \times 10^{-27} \text{ kg}$  and  $3.29 \times 10^{-25} \text{ kg}$ , respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of  $120^\circ$  during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

---

**Solution:**

(a)  $5.36 \times 10^5 \text{ m/s}$  at  $-29.5^\circ$

(b)  $7.52 \times 10^{-13} \text{ J}$

**Exercise:**

**Problem: Professional Application**

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at 8.00 m/s due south. The second car has a mass of 850 kg and is approaching at 17.0 m/s due west. (a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the  $x$ -axis and  $y$ -axis; instead, you must look for other simplifying aspects.

**Exercise:**

**Problem:**

Starting with equations  $m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$  and  $0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$  for conservation of momentum in the  $x$ - and  $y$ -directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

**Equation:**

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v'^2_1 + \frac{1}{2} m v'^2_2 + m v'_1 v'_2 \cos (\theta_1 - \theta_2)$$

as discussed in the text.

---

**Solution:**

We are given that  $m_1 = m_2 \equiv m$ . The given equations then become:

**Equation:**

$$v_1 = v'_1 \cos \theta_1 + v'_2 \cos \theta_2$$

and

**Equation:**

$$0 = v'_1 \sin \theta_1 + v'_2 \sin \theta_2.$$

Square each equation to get

**Equation:**

$$\begin{aligned}
 v_1^2 &= v_1'^2 \cos^2 \theta_1 + v_2'^2 \cos^2 \theta_2 + 2v_1'v_2' \cos \theta_1 \cos \theta_2 \\
 0 &= v_1'^2 \sin^2 \theta_1 + v_2'^2 \sin^2 \theta_2 + 2v_1'v_2' \sin \theta_1 \sin \theta_2.
 \end{aligned}$$

Add these two equations and simplify:

**Equation:**

$$\begin{aligned}
 v_1^2 &= v_1'^2 + v_2'^2 + 2v_1'v_2'(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\
 &= v_1'^2 + v_2'^2 + 2v_1'v_2' \left[ \frac{1}{2} \cos (\theta_1 - \theta_2) + \frac{1}{2} \cos (\theta_1 + \theta_2) + \frac{1}{2} \cos (\theta_1 - \theta_2) - \frac{1}{2} \cos (\theta_1 + \theta_2) \right] \\
 &= v_1'^2 + v_2'^2 + 2v_1'v_2' \cos (\theta_1 - \theta_2).
 \end{aligned}$$

Multiply the entire equation by  $\frac{1}{2}m$  to recover the kinetic energy:

**Equation:**

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv_1'v_2' \cos(\theta_1 - \theta_2)$$

**Exercise:**

### **Problem: Integrated Concepts**

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

## **Glossary**

point masses

structureless particles with no rotation or spin

## Introduction to Rocket Propulsion

- State Newton's third law of motion.
- Explain the principle involved in propulsion of rockets and jet engines.
- Derive an expression for the acceleration of the rocket and discuss the factors that affect the acceleration.
- Describe the function of a space shuttle.

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton's third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun's recoil or kick.

### **Note:**

#### **Making Connections: Take-Home Experiment—Propulsion of a Balloon**

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon's direction change? Explain your answer.

[\[link\]](#) shows a rocket accelerating straight up. In part (a), the rocket has a mass  $m$  and a velocity  $v$  relative to Earth, and hence a momentum  $mv$ . In part (b), a time  $\Delta t$  has elapsed in which the rocket has ejected a mass  $\Delta m$  of hot gas at a velocity  $v_e$  relative to the rocket. The remainder of the mass ( $m - \Delta m$ ) now has a greater velocity ( $v + \Delta v$ ). The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time  $\Delta t$ , producing a negative impulse  $\Delta p = -mg\Delta t$ . (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum



of the system.) So, the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket's thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.

By calculating the change in momentum for the entire system over  $\Delta t$ , and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

**Equation:**

$$a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$$

“The rocket” is that part of the system remaining after the gas is ejected, and  $g$  is the acceleration due to gravity.

**Note:**

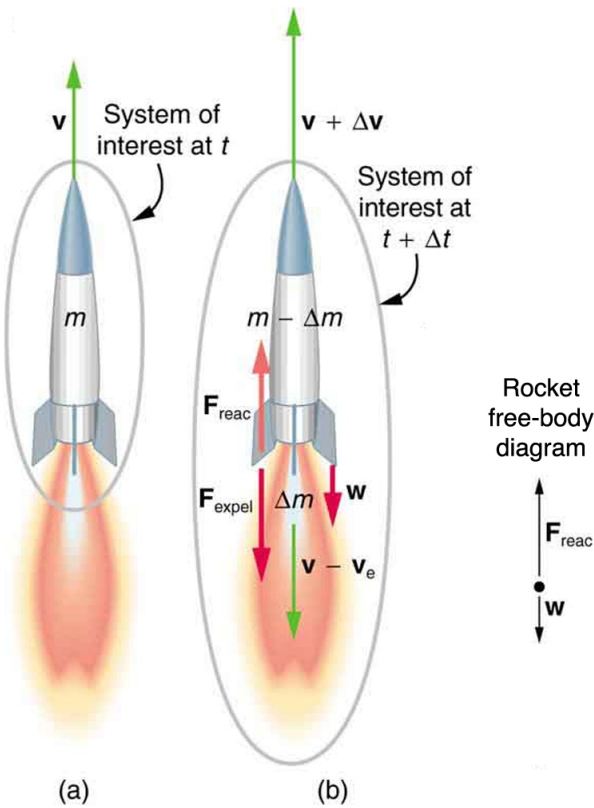
Acceleration of a Rocket

Acceleration of a rocket is

**Equation:**

$$a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g,$$

where  $a$  is the acceleration of the rocket,  $v_e$  is the exhaust velocity,  $m$  is the mass of the rocket,  $\Delta m$  is the mass of the ejected gas, and  $\Delta t$  is the time in which the gas is ejected.



(a) This rocket has a mass  $m$  and an upward velocity  $v$ . The net external force on the system is  $-mg$ , if air resistance is neglected. (b) A time  $\Delta t$  later the system has two main parts, the ejected gas and the remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.

A rocket's acceleration depends on three major factors, consistent with the equation for acceleration of a rocket. First, the greater the exhaust velocity of the gases relative to the rocket,  $v_e$ , the greater the acceleration is. The

practical limit for  $v_e$  is about  $2.5 \times 10^3$  m/s for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor  $\Delta m / \Delta t$  in the equation. The quantity  $(\Delta m / \Delta t)v_e$ , with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass  $m$  of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass  $m$  decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

**Note:****Factors Affecting a Rocket's Acceleration**

- The greater the exhaust velocity  $v_e$  of the gases relative to the rocket, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket's mass (all other factors being the same), the greater the acceleration.

**Example:****Calculating Acceleration: Initial Acceleration of a Moon Launch**

A Saturn V's mass at liftoff was  $2.80 \times 10^6$  kg, its fuel-burn rate was  $1.40 \times 10^4$  kg/s, and the exhaust velocity was  $2.40 \times 10^3$  m/s. Calculate its initial acceleration.

**Strategy**

This problem is a straightforward application of the expression for acceleration because  $a$  is the unknown and all of the terms on the right side of the equation are given.

**Solution**

Substituting the given values into the equation for acceleration yields

**Equation:**

$$\begin{aligned}
 a &= \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g \\
 &= \frac{2.40 \times 10^3 \text{ m/s}}{2.80 \times 10^6 \text{ kg}} (1.40 \times 10^4 \text{ kg/s}) - 9.80 \text{ m/s}^2 \\
 &= 2.20 \text{ m/s}^2.
 \end{aligned}$$

### Discussion

This value is fairly small, even for an initial acceleration. The acceleration does increase steadily as the rocket burns fuel, because  $m$  decreases while  $v_e$  and  $\frac{\Delta m}{\Delta t}$  remain constant. Knowing this acceleration and the mass of the rocket, you can show that the thrust of the engines was  $3.36 \times 10^7 \text{ N}$ .

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

**Equation:**

$$v = v_e \ln \frac{m_0}{m_r},$$

where  $\ln(m_0/m_r)$  is the natural logarithm of the ratio of the initial mass of the rocket ( $m_0$ ) to what is left ( $m_r$ ) after all of the fuel is exhausted. (Note that  $v$  is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.) For example, let us calculate the mass ratio needed to escape Earth's gravity starting from rest, given that the escape velocity from Earth is about  $11.2 \times 10^3 \text{ m/s}$ , and assuming an exhaust velocity  $v_e = 2.5 \times 10^3 \text{ m/s}$ .

**Equation:**

$$\ln \frac{m_0}{m_r} = \frac{v}{v_e} = \frac{11.2 \times 10^3 \text{ m/s}}{2.5 \times 10^3 \text{ m/s}} = 4.48$$

Solving for  $m_0/m_r$  gives

**Equation:**

$$\frac{m_0}{m_r} = e^{4.48} = 88.$$

Thus, the mass of the rocket is

**Equation:**

$$m_r = \frac{m_0}{88}.$$

This result means that only 1/88 of the mass is left when the fuel is burnt, and 87/88 of the initial mass was fuel. Expressed as percentages, 98.9% of the rocket is fuel, while payload, engines, fuel tanks, and other components make up only 1.10%. Taking air resistance and gravitational force into account, the mass  $m_r$  remaining can only be about  $m_0/180$ . It is difficult to build a rocket in which the fuel has a mass 180 times everything else. The solution is multistage rockets. Each stage only needs to achieve part of the final velocity and is discarded after it burns its fuel. The result is that each successive stage can have smaller engines and more payload relative to its fuel. Once out of the atmosphere, the ratio of payload to fuel becomes more favorable, too.

The space shuttle was an attempt at an economical vehicle with some reusable parts, such as the solid fuel boosters and the craft itself. (See [\[link\]](#)) The shuttle's need to be operated by humans, however, made it at least as costly for launching satellites as expendable, unmanned rockets. Ideally, the shuttle would only have been used when human activities were required for the success of a mission, such as the repair of the Hubble space telescope. Rockets with satellites can also be launched from airplanes. Using airplanes has the double advantage that the initial velocity is significantly above zero and a rocket can avoid most of the atmosphere's resistance.



The space shuttle had a number of reusable parts.

Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space shuttle was a complex assemblage of technologies, employing both solid and liquid fuel and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to

single-use rockets.  
(credit: NASA)

**Note:**

**PhET Explorations: Lunar Lander**

Can you avoid the boulder field and land safely, just before your fuel runs out, as Neil Armstrong did in 1969? Our version of this classic video game accurately simulates the real motion of the lunar lander with the correct mass, thrust, fuel consumption rate, and lunar gravity. The real lunar lander is very hard to control.

[https://phet.colorado.edu/sims/lunar-lander/lunar-lander\\_en.html](https://phet.colorado.edu/sims/lunar-lander/lunar-lander_en.html)

## Section Summary

- Newton's third law of motion states that to every action, there is an equal and opposite reaction.
- Acceleration of a rocket is  $a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$ .
- A rocket's acceleration depends on three main factors. They are
  1. The greater the exhaust velocity of the gases, the greater the acceleration.
  2. The faster the rocket burns its fuel, the greater its acceleration.
  3. The smaller the rocket's mass, the greater the acceleration.

## Conceptual Questions

**Exercise:**

**Problem: Professional Application**

Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How is the motion of the center

of mass affected by the explosion? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

**Exercise:**

**Problem: Professional Application**

During a visit to the International Space Station, an astronaut was positioned motionless in the center of the station, out of reach of any solid object on which he could exert a force. Suggest a method by which he could move himself away from this position, and explain the physics involved.

**Exercise:**

**Problem: Professional Application**

It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

## **Problems & Exercises**

**Exercise:**

**Problem: Professional Application**

Antiballistic missiles (ABMs) are designed to have very large accelerations so that they may intercept fast-moving incoming missiles in the short time available. What is the takeoff acceleration of a 10,000-kg ABM that expels 196 kg of gas per second at an exhaust velocity of  $2.50 \times 10^3 \text{ m/s}$ ?

---

**Solution:**



$$39.2 \text{ m/s}^2$$

**Exercise:**

**Problem: Professional Application**

What is the acceleration of a 5000-kg rocket taking off from the Moon, where the acceleration due to gravity is only  $1.6 \text{ m/s}^2$ , if the rocket expels 8.00 kg of gas per second at an exhaust velocity of  $2.20 \times 10^3 \text{ m/s}$ ?

**Exercise:**

**Problem: Professional Application**

Calculate the increase in velocity of a 4000-kg space probe that expels 3500 kg of its mass at an exhaust velocity of  $2.00 \times 10^3 \text{ m/s}$ . You may assume the gravitational force is negligible at the probe's location.

---

**Solution:**

$$4.16 \times 10^3 \text{ m/s}$$

**Exercise:**

**Problem: Professional Application**

Ion-propulsion rockets have been proposed for use in space. They employ atomic ionization techniques and nuclear energy sources to produce extremely high exhaust velocities, perhaps as great as  $8.00 \times 10^6 \text{ m/s}$ . These techniques allow a much more favorable payload-to-fuel ratio. To illustrate this fact: (a) Calculate the increase in velocity of a 20,000-kg space probe that expels only 40.0-kg of its mass at the given exhaust velocity. (b) These engines are usually designed to produce a very small thrust for a very long time—the type of engine that might be useful on a trip to the outer planets, for example. Calculate the acceleration of such an engine if it expels

$4.50 \times 10^{-6} \text{ kg/s}$  at the given velocity, assuming the acceleration due to gravity is negligible.

**Exercise:**

**Problem:** Derive the equation for the vertical acceleration of a rocket.

---

**Solution:**

The force needed to give a small mass  $\Delta m$  an acceleration  $a_{\Delta m}$  is  $F = \Delta m a_{\Delta m}$ . To accelerate this mass in the small time interval  $\Delta t$  at a speed  $v_e$  requires  $v_e = a_{\Delta m} \Delta t$ , so  $F = v_e \frac{\Delta m}{\Delta t}$ . By Newton's third law, this force is equal in magnitude to the thrust force acting on the rocket, so  $F_{\text{thrust}} = v_e \frac{\Delta m}{\Delta t}$ , where all quantities are positive. Applying Newton's second law to the rocket gives

$F_{\text{thrust}} - mg = ma \Rightarrow a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$ , where  $m$  is the mass of the rocket and unburnt fuel.

**Exercise:**

**Problem: Professional Application**

(a) Calculate the maximum rate at which a rocket can expel gases if its acceleration cannot exceed seven times that of gravity. The mass of the rocket just as it runs out of fuel is 75,000-kg, and its exhaust velocity is  $2.40 \times 10^3 \text{ m/s}$ . Assume that the acceleration of gravity is the same as on Earth's surface ( $9.80 \text{ m/s}^2$ ). (b) Why might it be necessary to limit the acceleration of a rocket?

**Exercise:**

**Problem:**

Given the following data for a fire extinguisher-toy wagon rocket experiment, calculate the average exhaust velocity of the gases expelled from the extinguisher. Starting from rest, the final velocity is 10.0 m/s. The total mass is initially 75.0 kg and is 70.0 kg after the extinguisher is fired.

**Exercise:****Problem:**

How much of a single-stage rocket that is 100,000 kg can be anything but fuel if the rocket is to have a final speed of 8.00 km/s, given that it expels gases at an exhaust velocity of  $2.20 \times 10^3$  m/s?

---

**Solution:**

$$2.63 \times 10^3 \text{ kg}$$

**Exercise:****Problem: Professional Application**

(a) A 5.00-kg squid initially at rest ejects 0.250-kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement. (b) How much energy is lost to work done against friction?

---

**Solution:**

(a) 0.421 m/s away from the ejected fluid.

(b) 0.237 J.

**Exercise:****Problem: Unreasonable Results**

Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of  $20.0^\circ$ , assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

**Exercise:**

**Problem: Construct Your Own Problem**

Consider an astronaut in deep space cut free from her space ship and needing to get back to it. The astronaut has a few packages that she can throw away to move herself toward the ship. Construct a problem in which you calculate the time it takes her to get back by throwing all the packages at one time compared to throwing them one at a time. Among the things to be considered are the masses involved, the force she can exert on the packages through some distance, and the distance to the ship.

**Exercise:**

**Problem: Construct Your Own Problem**

Consider an artillery projectile striking armor plating. Construct a problem in which you find the force exerted by the projectile on the plate. Among the things to be considered are the mass and speed of the projectile and the distance over which its speed is reduced. Your instructor may also wish for you to consider the relative merits of depleted uranium versus lead projectiles based on the greater density of uranium.

## Introduction to Rotational Motion and Angular Momentum

class="introduction"

The mention of  
a tornado  
conjures up  
images of raw  
destructive  
power.

Tornadoes  
blow houses  
away as if they  
were made of  
paper and have  
been known to  
pierce tree  
trunks with  
pieces of straw.

They descend  
from clouds in  
funnel-like  
shapes that spin  
violently,  
particularly at  
the bottom  
where they are  
most narrow,  
producing  
winds as high  
as 500 km/h.

(credit: Daphne  
Zaras, U.S.

National  
Oceanic and  
Atmospheric  
Administration

)



Why do tornadoes spin at all? And why do tornados spin so rapidly? The answer is that air masses that produce tornadoes are themselves rotating, and when the radii of the air masses decrease, their rate of rotation increases. An ice skater increases her spin in an exactly analogous manner as seen in [\[link\]](#). The skater starts her rotation with outstretched limbs and increases her spin by pulling them in toward her body. The same physics describes the exhilarating spin of a skater and the wrenching force of a tornado.

Clearly, force, energy, and power are associated with rotational motion. These and other aspects of rotational motion are covered in this chapter. We shall see that all important aspects of rotational motion either have already been defined for linear motion or have exact analogs in linear motion. First, we look at angular acceleration—the rotational analog of linear acceleration.



This figure skater increases her rate of spin by pulling her arms and her extended leg closer to her axis of rotation.  
(credit: Luu, Wikimedia Commons)

## Rotation Angle and Angular Velocity

- Define arc length, rotation angle, radius of curvature and angular velocity.
- Calculate the angular velocity of a car wheel spin.

In [Kinematics](#), we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. [Two-Dimensional Kinematics](#) dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

### Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in [\[link\]](#) rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle**  $\Delta\theta$  to be the ratio of the arc length to the radius of curvature:

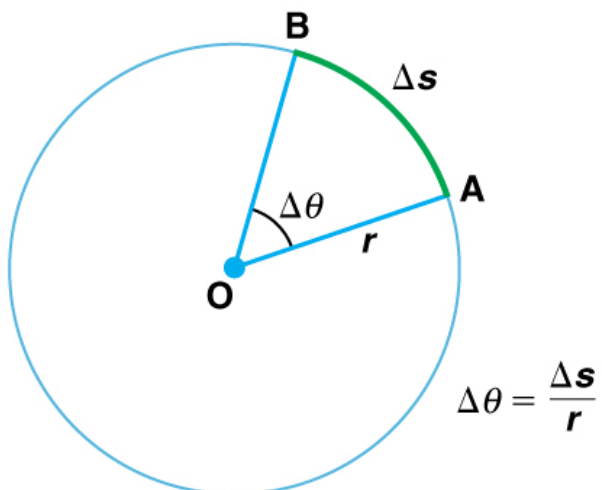
**Equation:**

$$\Delta\theta = \frac{\Delta s}{r}.$$





All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle  $\Delta\theta$  in a time  $\Delta t$ .



The radius of a circle is rotated through an angle  $\Delta\theta$ . The arc

length  $\Delta s$  is described on the circumference.

The **arc length**  $\Delta s$  is the distance traveled along a circular path as shown in [\[link\]](#) Note that  $r$  is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius  $r$ . The circumference of a circle is  $2\pi r$ . Thus for one complete revolution the rotation angle is

**Equation:**

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi.$$

This result is the basis for defining the units used to measure rotation angles,  $\Delta\theta$  to be **radians** (rad), defined so that

**Equation:**

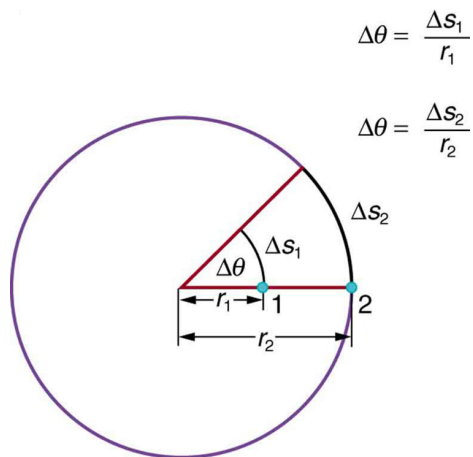
$$2\pi \text{ rad} = 1 \text{ revolution.}$$

A comparison of some useful angles expressed in both degrees and radians is shown in [\[link\]](#).

Degree Measures	Radian Measure
$30^\circ$	$\frac{\pi}{6}$

Degree Measures	Radian Measure
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	$\pi$

Comparison of Angular Units



Points 1 and 2 rotate through the same angle ( $\Delta\theta$ ), but point 2 moves through a greater arc length ( $\Delta s$ ) because it is at a greater distance from the center of rotation ( $r$ ).

If  $\Delta\theta = 2\pi$  rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are  $360^\circ$  in a circle or one revolution, the relationship between radians and degrees is thus

**Equation:**

$$2\pi \text{ rad} = 360^\circ$$

so that

**Equation:**

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ.$$

## Angular Velocity

How fast is an object rotating? We define **angular velocity**  $\omega$  as the rate of change of an angle. In symbols, this is

**Equation:**

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where an angular rotation  $\Delta\theta$  takes place in a time  $\Delta t$ . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity  $\omega$  is analogous to linear velocity  $v$ . To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length  $\Delta s$  in a time  $\Delta t$ , and so it has a linear velocity

**Equation:**

$$v = \frac{\Delta s}{\Delta t}.$$

From  $\Delta\theta = \frac{\Delta s}{r}$  we see that  $\Delta s = r\Delta\theta$ . Substituting this into the expression for  $v$  gives

**Equation:**

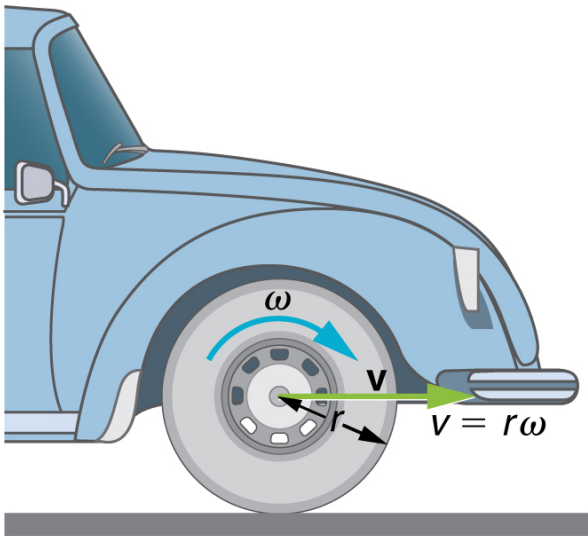
$$v = \frac{r\Delta\theta}{\Delta t} = r\omega.$$

We write this relationship in two different ways and gain two different insights:

**Equation:**

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

The first relationship in  $v = r\omega$  or  $\omega = \frac{v}{r}$  states that the linear velocity  $v$  is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest  $r$ ), as you might expect. We can also call this linear speed  $v$  of a point on the rim the *tangential speed*. The second relationship in  $v = r\omega$  or  $\omega = \frac{v}{r}$  can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed  $v$  of the car. See [\[link\]](#). So the faster the car moves, the faster the tire spins—large  $v$  means a large  $\omega$ , because  $v = r\omega$ . Similarly, a larger-radius tire rotating at the same angular velocity ( $\omega$ ) will produce a greater linear speed ( $v$ ) for the car.



A car moving at a velocity  $v$  to the right has a tire rotating with an angular velocity  $\omega$ . The speed of the tread of the tire relative to the axle is  $v$ , the same as if the car were jacked up. Thus the car moves forward at linear velocity  $v = r\omega$ , where  $r$  is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

**Example:****How Fast Does a Car Tire Spin?**

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See [\[link\]](#).

**Strategy**

Because the linear speed of the tire rim is the same as the speed of the car, we have  $v = 15.0$  m/s. The radius of the tire is given to be  $r = 0.300$  m. Knowing  $v$  and  $r$ , we can use the second relationship in  $v = r\omega$ ,  $\omega = \frac{v}{r}$  to calculate the angular velocity.

**Solution**

To calculate the angular velocity, we will use the following relationship:

**Equation:**

$$\omega = \frac{v}{r}.$$

Substituting the knowns,

**Equation:**

$$\omega = \frac{15.0 \text{ m/s}}{0.300 \text{ m}} = 50.0 \text{ rad/s}.$$

**Discussion**

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

**Equation:**

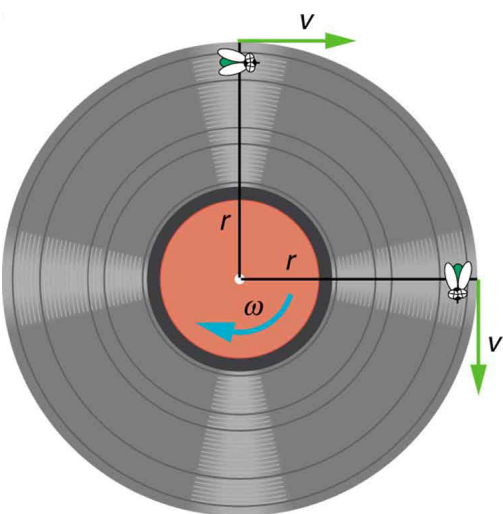
$$\omega = (15.0 \text{ m/s})/(1.20 \text{ m}) = 12.5 \text{ rad/s}.$$

Both  $\omega$  and  $v$  have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in [\[link\]](#).

**Note:**

**Take-Home Experiment**

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.



As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is



always tangent to the circle. The direction of the angular velocity is clockwise in this case.

**Note:**

PhET Explorations: Ladybug Revolution

[Ladybug  
Revolutio  
n](#)

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

## Section Summary

- Uniform circular motion is motion in a circle at constant speed. The rotation angle  $\Delta\theta$  is defined as the ratio of the arc length to the radius of curvature:

**Equation:**

$$\Delta\theta = \frac{\Delta s}{r},$$

where arc length  $\Delta s$  is distance traveled along a circular path and  $r$  is the radius of curvature of the circular path. The quantity  $\Delta\theta$  is

measured in units of radians (rad), for which

**Equation:**

$$2\pi \text{ rad} = 360^\circ = 1 \text{ revolution.}$$

- The conversion between radians and degrees is  $1 \text{ rad} = 57.3^\circ$ .
- Angular velocity  $\omega$  is the rate of change of an angle,

**Equation:**

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where a rotation  $\Delta\theta$  takes place in a time  $\Delta t$ . The units of angular velocity are radians per second (rad/s). Linear velocity  $v$  and angular velocity  $\omega$  are related by

**Equation:**

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

## Conceptual Questions

**Exercise:**

**Problem:**

There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

## Problem Exercises

**Exercise:**

**Problem:**

Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200,000 rotations, how many kilometers should the odometer read?

---

**Solution:**

723 km

**Exercise:****Problem:**

Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

**Exercise:****Problem:**

An automobile with 0.260 m radius tires travels 80,000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

---

**Solution:**

$5 \times 10^7$  rotations

**Exercise:****Problem:**

(a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of  $6.4 \times 10^6$  m at its equator, what is the linear velocity at Earth's surface?

**Exercise:****Problem:**

A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher's hand is  $35.0 \text{ m/s}$  and the ball is  $0.300 \text{ m}$  from the elbow joint, what is the angular velocity of the forearm?

---

**Solution:**

$117 \text{ rad/s}$

**Exercise:****Problem:**

In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is  $30.0 \text{ rad/s}$  and the ball is  $1.30 \text{ m}$  from the elbow joint, what is the velocity of the ball?

**Exercise:****Problem:**

A truck with  $0.420\text{-m}$ -radius tires travels at  $32.0 \text{ m/s}$ . What is the angular velocity of the rotating tires in radians per second? What is this in  $\text{rev/min}$ ?

---

**Solution:**

$76.2 \text{ rad/s}$

$728 \text{ rpm}$

**Exercise:****Problem:**

**Integrated Concepts** When kicking a football, the kicker rotates his leg about the hip joint.

- (a) If the velocity of the tip of the kicker's shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip's angular velocity?
- (b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?
- (c) Find the maximum range of the football, neglecting air resistance.
- 

**Solution:**

- (a) 33.3 rad/s
- (b) 500 N
- (c) 40.8 m

**Exercise:**

**Problem:Construct Your Own Problem**

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders' clothing and the wall.

**Glossary**

arc length

$\Delta s$ , the distance traveled by an object along a circular path

pit

a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

rotation angle

the ratio of the arc length to the radius of curvature on a circular path:

$$\Delta\theta = \frac{\Delta s}{r}$$

radius of curvature

radius of a circular path

radians

a unit of angle measurement

angular velocity

$\omega$ , the rate of change of the angle with which an object moves on a circular path

## Angular Acceleration

- Describe uniform circular motion.
- Explain non-uniform circular motion.
- Calculate angular acceleration of an object.
- Observe the link between linear and angular acceleration.

[Uniform Circular Motion and Gravitation](#) discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity  $\omega$  was defined as the time rate of change of angle  $\theta$ :

**Equation:**

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where  $\theta$  is the angle of rotation as seen in [\[link\]](#). The relationship between angular velocity  $\omega$  and linear velocity  $v$  was also defined in [Rotation Angle and Angular Velocity](#) as

**Equation:**

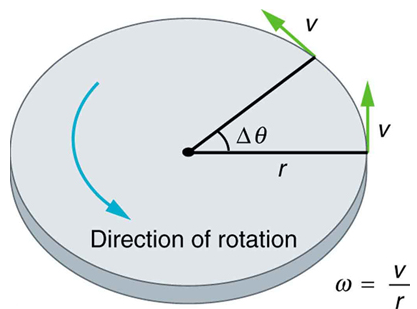
$$v = r\omega$$

or

**Equation:**

$$\omega = \frac{v}{r},$$

where  $r$  is the radius of curvature, also seen in [\[link\]](#). According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative



This figure shows uniform circular motion and some of its defined quantities.

Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there is an **angular acceleration**, in which  $\omega$  changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration  $\alpha$  is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

**Equation:**

$$\alpha = \frac{\Delta\omega}{\Delta t},$$

where  $\Delta\omega$  is the **change in angular velocity** and  $\Delta t$  is the change in time. The units of angular acceleration are (rad/s)/s, or  $\text{rad/s}^2$ . If  $\omega$  increases, then  $\alpha$  is positive. If  $\omega$  decreases, then  $\alpha$  is negative.

**Example:**

**Calculating the Angular Acceleration and Deceleration of a Bike Wheel**



Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in  $\text{rad/s}^2$ . (b) If she now slams on the brakes, causing an angular acceleration of  $-87.3 \text{ rad/s}^2$ , how long does it take the wheel to stop?

**Strategy for (a)**

The angular acceleration can be found directly from its definition in  $\alpha = \frac{\Delta\omega}{\Delta t}$  because the final angular velocity and time are given. We see that  $\Delta\omega$  is 250 rpm and  $\Delta t$  is 5.00 s.

**Solution for (a)**

Entering known information into the definition of angular acceleration, we get

**Equation:**

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{250 \text{ rpm}}{5.00 \text{ s}}.\end{aligned}$$

Because  $\Delta\omega$  is in revolutions per minute (rpm) and we want the standard units of  $\text{rad/s}^2$  for angular acceleration, we need to convert  $\Delta\omega$  from rpm to  $\text{rad/s}$ :

**Equation:**

$$\begin{aligned}\Delta\omega &= 250 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \\ &= 26.2 \frac{\text{rad}}{\text{s}}.\end{aligned}$$

Entering this quantity into the expression for  $\alpha$ , we get

**Equation:**

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{26.2 \text{ rad/s}}{5.00 \text{ s}} \\ &= 5.24 \text{ rad/s}^2.\end{aligned}$$

**Strategy for (b)**

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular acceleration and solving for  $\Delta t$ , yielding

**Equation:**

$$\Delta t = \frac{\Delta \omega}{\alpha}.$$

**Solution for (b)**

Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that  $\Delta \omega$  is  $-26.2$  rad/s, and  $\alpha$  is given to be  $-87.3$  rad/s<sup>2</sup>. Thus,

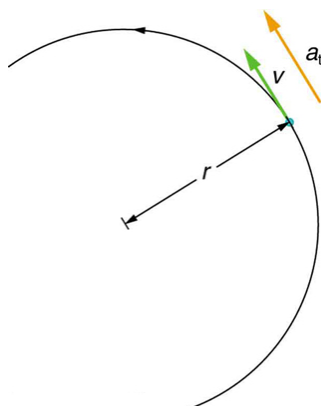
**Equation:**

$$\begin{aligned}\Delta t &= \frac{-26.2 \text{ rad/s}}{-87.3 \text{ rad/s}^2} \\ &= 0.300 \text{ s}.\end{aligned}$$

**Discussion**

Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

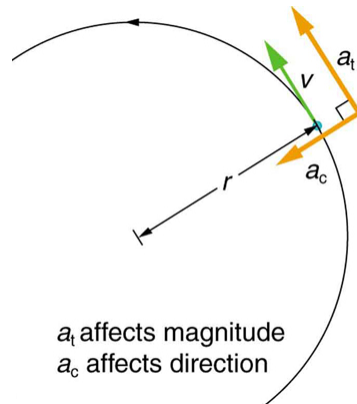
If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is *tangent* to the circle at the point of interest, as seen in [\[link\]](#). Thus, linear acceleration is called **tangential acceleration**  $a_t$ .



In circular motion, linear acceleration  $a$ , occurs as the magnitude of the velocity changes:  $a$  is tangent to the motion. In the context of circular motion, linear acceleration is also called tangential acceleration  $a_t$ .

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from [Uniform Circular Motion and Gravitation](#) that in circular motion centripetal acceleration,  $a_c$ , refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in [\[link\]](#). Thus,  $a_t$  and  $a_c$  are perpendicular and independent of one another.

Tangential acceleration  $a_t$  is directly related to the angular acceleration  $\alpha$  and is linked to an increase or decrease in the velocity, but not its direction.



Centripetal acceleration  $a_c$  occurs as the direction of velocity changes; it is perpendicular to the circular motion.

Centripetal and tangential acceleration are thus perpendicular to each other.

Now we can find the exact relationship between linear acceleration  $a_t$  and angular acceleration  $\alpha$ . Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in [One-Dimensional Kinematics](#)) to be

**Equation:**

$$a_t = \frac{\Delta v}{\Delta t}.$$

For circular motion, note that  $v = r\omega$ , so that

**Equation:**

$$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$

The radius  $r$  is constant for circular motion, and so  $\Delta(r\omega) = r(\Delta\omega)$ . Thus,

**Equation:**

$$a_t = r \frac{\Delta\omega}{\Delta t}.$$

By definition,  $\alpha = \frac{\Delta\omega}{\Delta t}$ . Thus,

**Equation:**

$$a_t = r\alpha,$$

or

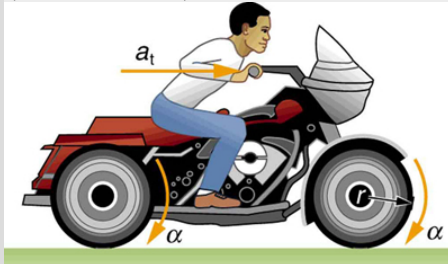
**Equation:**

$$\alpha = \frac{a_t}{r}.$$

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration  $\alpha$ .

**Example:****Calculating the Angular Acceleration of a Motorcycle Wheel**

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels? (See [\[link\]](#).)



The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.

**Strategy**

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration  $a_t$ . Then, the expression  $\alpha = \frac{a_t}{r}$  can be used to find the angular acceleration.

**Solution**

The linear acceleration is

**Equation:**

$$\begin{aligned} a_t &= \frac{\Delta v}{\Delta t} \\ &= \frac{30.0 \text{ m/s}}{4.20 \text{ s}} \\ &= 7.14 \text{ m/s}^2. \end{aligned}$$

We also know the radius of the wheels. Entering the values for  $a_t$  and  $r$  into  $\alpha = \frac{a_t}{r}$ , we get

**Equation:**

$$\begin{aligned}
 \alpha &= \frac{a_t}{r} \\
 &= \frac{7.14 \text{ m/s}^2}{0.320 \text{ m}} \\
 &= 22.3 \text{ rad/s}^2.
 \end{aligned}$$

### Discussion

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities—  $\theta$ ,  $\omega$ , and  $\alpha$ . These quantities are analogous to the translational quantities  $x$ ,  $v$ , and  $a$ . [\[link\]](#) displays rotational quantities, the analogous translational quantities, and the relationships between them.

Rotational	Translational	Relationship
$\theta$	$x$	$\theta = \frac{x}{r}$
$\omega$	$v$	$\omega = \frac{v}{r}$
$\alpha$	$a$	$\alpha = \frac{a_t}{r}$

Rotational and Translational Quantities

**Note:****Making Connections: Take-Home Experiment**

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

**Exercise:****Check Your Understanding****Problem:**

Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

**Solution:**

The magnitude of angular acceleration is  $\alpha$  and its most common units are  $\text{rad/s}^2$ . The direction of angular acceleration along a fixed axis is denoted by a + or a – sign, just as the direction of linear acceleration in one dimension is denoted by a + or a – sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

**Note:****PhET Explorations: Ladybug Revolution**

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or



angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

[Ladybug](#)  
[Revolutio](#)  
[n](#)

## Section Summary

- Uniform circular motion is the motion with a constant angular velocity  $\omega = \frac{\Delta\theta}{\Delta t}$ .
- In non-uniform circular motion, the velocity changes with time and the rate of change of angular velocity (i.e. angular acceleration) is  $\alpha = \frac{\Delta\omega}{\Delta t}$ .
- Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction, given as  $a_t = \frac{\Delta v}{\Delta t}$ .
- For circular motion, note that  $v = r\omega$ , so that

**Equation:**

$$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$

- The radius  $r$  is constant for circular motion, and so  $\Delta(r\omega) = r\Delta\omega$ . Thus,

**Equation:**

$$a_t = r \frac{\Delta\omega}{\Delta t}.$$

- By definition,  $\Delta\omega/\Delta t = \alpha$ . Thus,

**Equation:**

$$a_t = r\alpha$$

or

**Equation:**

$$\alpha = \frac{a_t}{r}.$$

## Conceptual Questions

**Exercise:**

**Problem:**

Analogies exist between rotational and translational physical quantities. Identify the rotational term analogous to each of the following: acceleration, force, mass, work, translational kinetic energy, linear momentum, impulse.

**Exercise:**

**Problem:**

Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

**Exercise:**

**Problem:**

In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

**Exercise:**

**Problem:**

Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) The plate starts to spin? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

**Problems & Exercises****Exercise:****Problem:**

At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?

---

**Solution:**

$$\omega = 0.737 \text{ rev/s}$$

**Exercise:****Problem: Integrated Concepts**

An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min.

(a) What is its angular acceleration in  $\text{rad/s}^2$ ? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the radial acceleration in  $\text{m/s}^2$  and multiples of  $g$  of this point at full rpm?

**Exercise:****Problem: Integrated Concepts**

You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction

between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

---

**Solution:**

(a)  $-0.26 \text{ rad/s}^2$

(b) 27 rev

**Exercise:**

**Problem: Unreasonable Results**

You are told that a basketball player spins the ball with an angular acceleration of  $100 \text{ rad/s}^2$ . (a) What is the ball's final angular velocity if the ball starts from rest and the acceleration lasts 2.00 s? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

**Glossary**

angular acceleration

the rate of change of angular velocity with time

change in angular velocity

the difference between final and initial values of angular velocity

tangential acceleration

the acceleration in a direction tangent to the circle at the point of interest in circular motion

## Kinematics of Rotational Motion

- Observe the kinematics of rotational motion.
- Derive rotational kinematic equations.
- Evaluate problem solving strategies for rotational kinematics.

Just by using our intuition, we can begin to see how rotational quantities like  $\theta$ ,  $\omega$ , and  $\alpha$  are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel's angular acceleration  $\alpha$  is large for a long period of time  $t$ , then the final angular velocity  $\omega$  and angle of rotation  $\theta$  are large. The wheel's rotational motion is exactly analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating  $\omega$ ,  $\alpha$ , and  $t$ . To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

**Equation:**

$$v = v_0 + at \quad (\text{constant } a)$$

Note that in rotational motion  $a = a_t$ , and we shall use the symbol  $a$  for tangential or linear acceleration from now on. As in linear kinematics, we assume  $a$  is constant, which means that angular acceleration  $\alpha$  is also a constant, because  $a = r\alpha$ . Now, let us substitute  $v = r\omega$  and  $a = r\alpha$  into the linear equation above:

**Equation:**

$$r\omega = r\omega_0 + r\alpha t.$$

The radius  $r$  cancels in the equation, yielding

**Equation:**

$$\omega = \omega_0 + at \quad (\text{constant } a),$$

where  $\omega_0$  is the initial angular velocity. This last equation is a *kinematic relationship* among  $\omega$ ,  $\alpha$ , and  $t$ —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its translational counterpart.

**Note:**

**Making Connections**

Kinematics for rotational motion is completely analogous to translational kinematics, first presented in [One-Dimensional Kinematics](#). Kinematics is concerned with the description of motion without regard to force or mass. We will find that translational kinematic quantities, such as displacement, velocity, and acceleration have direct analogs in rotational motion.

Starting with the four kinematic equations we developed in [One-Dimensional Kinematics](#), we can derive the following four rotational kinematic equations (presented together with their translational counterparts):

Rotational	Translational	
$\theta = \omega t$	$x = \bar{v}t$	

Rotational	Translational	
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	(constant $\alpha, a$ )
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$	(constant $\alpha, a$ )
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$	(constant $\alpha, a$ )

### Rotational Kinematic Equations

In these equations, the subscript 0 denotes initial values ( $\theta_0$ ,  $x_0$ , and  $t_0$  are initial values), and the average angular velocity  $\bar{\omega}$  and average velocity  $\bar{v}$  are defined as follows:

#### Equation:

$$\bar{\omega} = \frac{\omega_0 + \omega}{2} \text{ and } \bar{v} = \frac{v_0 + v}{2}.$$

The equations given above in [\[link\]](#) can be used to solve any rotational or translational kinematics problem in which  $a$  and  $\alpha$  are constant.

#### Note:

##### Problem-Solving Strategy for Rotational Kinematics

1. *Examine the situation to determine that rotational kinematics (rotational motion) is involved.* Rotation must be involved, but without the need to consider forces or masses that affect the motion.

2. *Identify exactly what needs to be determined in the problem (identify the unknowns). A sketch of the situation is useful.*
3. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns).*
4. *Solve the appropriate equation or equations for the quantity to be determined (the unknown). It can be useful to think in terms of a translational analog because by now you are familiar with such motion.*
5. *Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units. Be sure to use units of radians for angles.*
6. *Check your answer to see if it is reasonable: Does your answer make sense?*

### **Example:**

#### **Calculating the Acceleration of a Fishing Reel**

A deep-sea fisherman hooks a big fish that swims away from the boat pulling the fishing line from his fishing reel. The whole system is initially at rest and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of  $110 \text{ rad/s}^2$  for 2.00 s as seen in [\[link\]](#).

- (a) What is the final angular velocity of the reel?
- (b) At what speed is fishing line leaving the reel after 2.00 s elapses?
- (c) How many revolutions does the reel make?
- (d) How many meters of fishing line come off the reel in this time?

#### **Strategy**

In each part of this example, the strategy is the same as it was for solving problems in linear kinematics. In particular, known values are identified and a relationship is then sought that can be used to solve for the unknown.

#### **Solution for (a)**

Here  $\alpha$  and  $t$  are given and  $\omega$  needs to be determined. The most straightforward equation to use is  $\omega = \omega_0 + \alpha t$  because the unknown is already on one side and all other terms are known. That equation states that

#### **Equation:**



$$\omega = \omega_0 + \alpha t.$$

We are also given that  $\omega_0 = 0$  (it starts from rest), so that

**Equation:**

$$\omega = 0 + (110 \text{ rad/s}^2)(2.00\text{s}) = 220 \text{ rad/s}.$$

**Solution for (b)**

Now that  $\omega$  is known, the speed  $v$  can most easily be found using the relationship

**Equation:**

$$v = r\omega,$$

where the radius  $r$  of the reel is given to be 4.50 cm; thus,

**Equation:**

$$v = (0.0450 \text{ m})(220 \text{ rad/s}) = 9.90 \text{ m/s}.$$

Note again that radians must always be used in any calculation relating linear and angular quantities. Also, because radians are dimensionless, we have  $\text{m} \times \text{rad} = \text{m}$ .

**Solution for (c)**

Here, we are asked to find the number of revolutions. Because  $1 \text{ rev} = 2\pi \text{ rad}$ , we can find the number of revolutions by finding  $\theta$  in radians. We are given  $\alpha$  and  $t$ , and we know  $\omega_0$  is zero, so that  $\theta$  can be obtained using  $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ .

**Equation:**

$$\begin{aligned}\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= 0 + (0.500)\left(110 \text{ rad/s}^2\right)(2.00 \text{ s})^2 = 220 \text{ rad}.\end{aligned}$$

Converting radians to revolutions gives

**Equation:**

$$\theta = (220 \text{ rad})\frac{1 \text{ rev}}{2\pi \text{ rad}} = 35.0 \text{ rev}.$$

### Solution for (d)

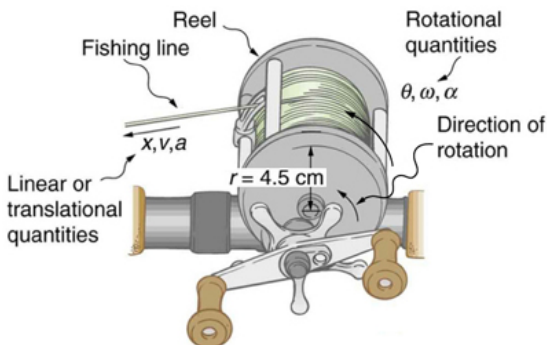
The number of meters of fishing line is  $x$ , which can be obtained through its relationship with  $\theta$ :

**Equation:**

$$x = r\theta = (0.0450 \text{ m})(220 \text{ rad}) = 9.90 \text{ m}.$$

### Discussion

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. We also see in this example how linear and rotational quantities are connected. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.) The amount of fishing line played out is 9.90 m, about right for when the big fish bites.



Fishing line coming off a rotating reel moves linearly. [\[link\]](#) and [\[link\]](#) consider relationships between rotational and linear quantities associated with a fishing reel.

**Example:****Calculating the Duration When the Fishing Reel Slows Down and Stops**

Now let us consider what happens if the fisherman applies a brake to the spinning reel, achieving an angular acceleration of  $-300 \text{ rad/s}^2$ . How long does it take the reel to come to a stop?

**Strategy**

We are asked to find the time  $t$  for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is  $\omega_0 = 220 \text{ rad/s}$  and the final angular velocity  $\omega$  is zero. The angular acceleration is given to be  $\alpha = -300 \text{ rad/s}^2$ . Examining the available equations, we see all quantities but  $t$  are known in  $\omega = \omega_0 + \alpha t$ , making it easiest to use this equation.

**Solution**

The equation states

**Equation:**

$$\omega = \omega_0 + \alpha t.$$

We solve the equation algebraically for  $t$ , and then substitute the known values as usual, yielding

**Equation:**

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 220 \text{ rad/s}}{-300 \text{ rad/s}^2} = 0.733 \text{ s}.$$

**Discussion**

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish will be slower, requiring a smaller acceleration.

**Example:****Calculating the Slow Acceleration of Trains and Their Wheels**

Large freight trains accelerate very slowly. Suppose one such train accelerates from rest, giving its 0.350-m-radius wheels an angular acceleration of  $0.250 \text{ rad/s}^2$ . After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

**Strategy**

In part (a), we are asked to find  $x$ , and in (b) we are asked to find  $\omega$  and  $v$ . We are given the number of revolutions  $\theta$ , the radius of the wheels  $r$ , and the angular acceleration  $\alpha$ .

**Solution for (a)**

The distance  $x$  is very easily found from the relationship between distance and rotation angle:

**Equation:**

$$\theta = \frac{x}{r}.$$

Solving this equation for  $x$  yields

**Equation:**

$$x = r\theta.$$

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

**Equation:**

$$\theta = (200 \text{ rev}) \frac{2\pi \text{ rad}}{1 \text{ rev}} = 1257 \text{ rad}.$$

Now we can substitute the known values into  $x = r\theta$  to find the distance the train moved down the track:

**Equation:**

$$x = r\theta = (0.350 \text{ m})(1257 \text{ rad}) = 440 \text{ m}.$$

**Solution for (b)**

We cannot use any equation that incorporates  $t$  to find  $\omega$ , because the equation would have at least two unknown values. The equation  $\omega^2 = \omega_0^2 + 2\alpha\theta$  will work, because we know the values for all variables except  $\omega$ :

**Equation:**

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Taking the square root of this equation and entering the known values gives

**Equation:**

$$\begin{aligned}\omega &= \left[ 0 + 2(0.250 \text{ rad/s}^2)(1257 \text{ rad}) \right]^{1/2} \\ &= 25.1 \text{ rad/s.}\end{aligned}$$

We can find the linear velocity of the train,  $v$ , through its relationship to  $\omega$ :

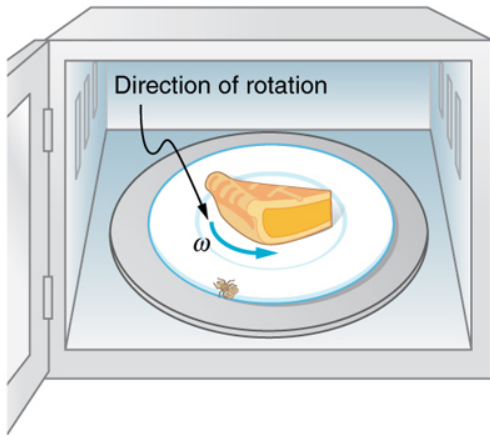
**Equation:**

$$v = r\omega = (0.350 \text{ m})(25.1 \text{ rad/s}) = 8.77 \text{ m/s.}$$

**Discussion**

The distance traveled is fairly large and the final velocity is fairly slow (just under 32 km/h).

There is translational motion even for something spinning in place, as the following example illustrates. [\[link\]](#) shows a fly on the edge of a rotating microwave oven plate. The example below calculates the total distance it travels.



The image shows a microwave plate. The fly makes revolutions while the food is heated (along with the fly).

**Example:****Calculating the Distance Traveled by a Fly on the Edge of a Microwave Oven Plate**

A person decides to use a microwave oven to reheat some lunch. In the process, a fly accidentally flies into the microwave and lands on the outer edge of the rotating plate and remains there. If the plate has a radius of 0.15 m and rotates at 6.0 rpm, calculate the total distance traveled by the fly during a 2.0-min cooking period. (Ignore the start-up and slow-down times.)

**Strategy**

First, find the total number of revolutions  $\theta$ , and then the linear distance  $x$  traveled.  $\theta = \omega t$  can be used to find  $\theta$  because  $\omega$  is given to be 6.0 rpm.

**Solution**

Entering known values into  $\theta = \omega t$  gives

**Equation:**

$$\theta = \omega t = (6.0 \text{ rpm})(2.0 \text{ min}) = 12 \text{ rev.}$$

As always, it is necessary to convert revolutions to radians before calculating a linear quantity like  $x$  from an angular quantity like  $\theta$ :

**Equation:**

$$\theta = (12 \text{ rev}) \left( \frac{2\pi \text{ rad}}{1 \text{ rev}} \right) = 75.4 \text{ rad.}$$

Now, using the relationship between  $x$  and  $\theta$ , we can determine the distance traveled:

**Equation:**

$$x = r\theta = (0.15 \text{ m})(75.4 \text{ rad}) = 11 \text{ m.}$$

### Discussion

Quite a trip (if it survives)! Note that this distance is the total distance traveled by the fly. Displacement is actually zero for complete revolutions because they bring the fly back to its original position. The distinction between total distance traveled and displacement was first noted in [One-Dimensional Kinematics](#).

### Exercise:

#### Check Your Understanding

##### Problem:

Rotational kinematics has many useful relationships, often expressed in equation form. Are these relationships laws of physics or are they simply descriptive? (Hint: the same question applies to linear kinematics.)

---

##### Solution:

Rotational kinematics (just like linear kinematics) is descriptive and does not represent laws of nature. With kinematics, we can describe

many things to great precision but kinematics does not consider causes. For example, a large angular acceleration describes a very rapid change in angular velocity without any consideration of its cause.

## Section Summary

- Kinematics is the description of motion.
- The kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time.
- Starting with the four kinematic equations we developed in the [One-Dimensional Kinematics](#), we can derive the four rotational kinematic equations (presented together with their translational counterparts) seen in [\[link\]](#).
- In these equations, the subscript 0 denotes initial values ( $x_0$  and  $t_0$  are initial values), and the average angular velocity  $\omega$  and average velocity  $\bar{v}$  are defined as follows:

**Equation:**

$$\omega = \frac{\omega_0 + \omega}{2} \text{ and } v = \frac{v_0 + v}{2}.$$

## Problems & Exercises

**Exercise:**

**Problem:**

With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s.

- (a) What is its angular acceleration in rad/s<sup>2</sup>?
- (b) How many revolutions does it go through in the process?

---

**Solution:**



(a)  $80 \text{ rad/s}^2$

(b) 1.0 rev

**Exercise:**

**Problem:**

Suppose a piece of dust finds itself on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

**Exercise:**

**Problem:**

A gyroscope slows from an initial rate of  $32.0 \text{ rad/s}$  at a rate of  $0.700 \text{ rad/s}^2$ .

(a) How long does it take to come to rest?

(b) How many revolutions does it make before stopping?

---

**Solution:**

(a) 45.7 s

(b) 116 rev

**Exercise:**

**Problem:** During a very quick stop, a car decelerates at  $7.00 \text{ m/s}^2$ .

(a) What is the angular acceleration of its 0.280-m-radius tires, assuming they do not slip on the pavement?

(b) How many revolutions do the tires make before coming to rest, given their initial angular velocity is  $95.0 \text{ rad/s}$ ?

- (c) How long does the car take to stop completely?
- (d) What distance does the car travel in this time?
- (e) What was the car's initial velocity?
- (f) Do the values obtained seem reasonable, considering that this stop happens very quickly?



Yo-yos are amusing toys that display significant physics and are engineered to enhance performance based on physical laws. (credit: Beyond Neon, Flickr)

### **Exercise:**

#### **Problem:**

Everyday application: Suppose a yo-yo has a center shaft that has a 0.250 cm radius and that its string is being pulled.

- (a) If the string is stationary and the yo-yo accelerates away from it at a rate of  $1.50 \text{ m/s}^2$ , what is the angular acceleration of the yo-yo?
- (b) What is the angular velocity after 0.750 s if it starts from rest?

(c) The outside radius of the yo-yo is 3.50 cm. What is the tangential acceleration of a point on its edge?

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**Solution:**

- a)  $600 \text{ rad/s}^2$
- b)  $450 \text{ rad/s}$
- c)  $21.0 \text{ m/s}$

**Glossary**

kinematics of rotational motion

describes the relationships among rotation angle, angular velocity, angular acceleration, and time

## Torque

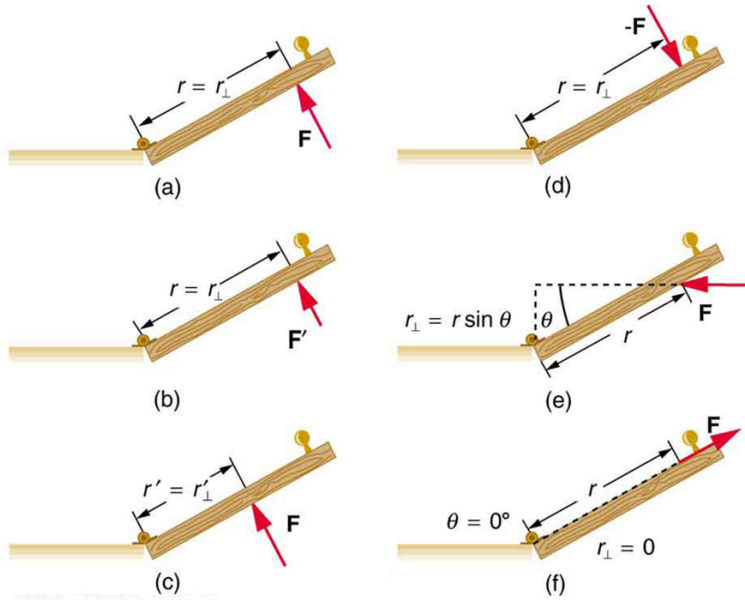
- State the second condition that is necessary to achieve equilibrium.
- Explain torque and the factors on which it depends.
- Describe the role of torque in rotational mechanics.

### **Note:**

#### Torque

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. To understand what factors affect rotation, let us think about what happens when you open an ordinary door by rotating it on its hinges.

Several familiar factors determine how effective you are in opening the door. See [\[link\]](#). First of all, the larger the force, the more effective it is in opening the door—obviously, the harder you push, the more rapidly the door opens. Also, the point at which you push is crucial. If you apply your force too close to the hinges, the door will open slowly, if at all. Most people have been embarrassed by making this mistake and bumping up against a door when it did not open as quickly as expected. Finally, the direction in which you push is also important. The most effective direction is perpendicular to the door—we push in this direction almost instinctively.



Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) Counterclockwise torque is produced by this force, which means that the door will rotate in a counterclockwise due to  $\mathbf{F}$ . Note that  $r_{\perp}$  is the perpendicular distance of the pivot from the line of action of the force. (b) A smaller counterclockwise torque is produced by a smaller force  $\mathbf{F}'$  acting at the same distance from the hinges (the pivot point). (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) The same force as in (a), but acting in the opposite direction, produces a clockwise torque. (e) A smaller counterclockwise torque is produced by the same magnitude force acting at the same point

but in a different direction. Here,  $\theta$  is less than  $90^\circ$ . (f) Torque is zero here since the force just pulls on the hinges, producing no rotation. In this case,  $\theta = 0^\circ$ .

The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called torque.

**Torque** is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

**Equation:**

$$\tau = rF \sin \theta$$

where  $\tau$  (the Greek letter tau) is the symbol for torque,  $r$  is the distance from the pivot point to the point where the force is applied,  $F$  is the magnitude of the force, and  $\theta$  is the angle between the force and the vector directed from the point of application to the pivot point, as seen in [\[link\]](#) and [\[link\]](#). An alternative expression for torque is given in terms of the **perpendicular lever arm**  $r_\perp$  as shown in [\[link\]](#) and [\[link\]](#), which is defined as

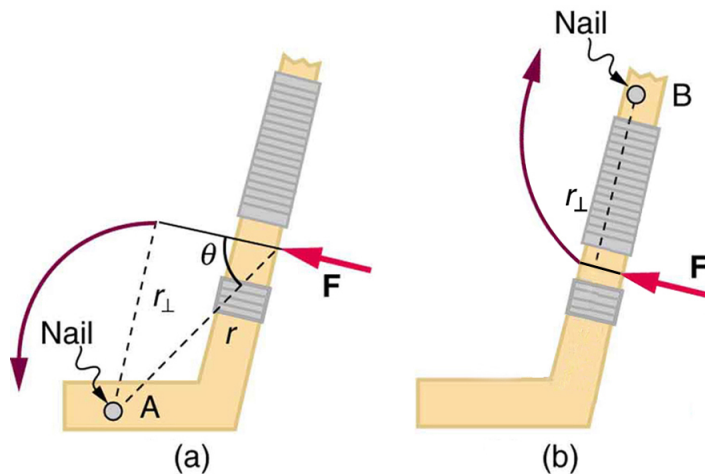
**Equation:**

$$r_\perp = r \sin \theta$$

so that

**Equation:**

$$\tau = r_\perp F.$$



A force applied to an object can produce a torque, which depends on the location of the pivot point. (a) The three factors  $r$ ,  $F$ , and  $\theta$  for pivot point A on a body are shown here— $r$  is the distance from the chosen pivot point to the point where the force  $F$  is applied, and  $\theta$  is the angle between  $\mathbf{F}$  and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A. (b) In this case, point B is the pivot point. The torque from the applied force will cause a clockwise rotation around point B, and so it is a clockwise torque relative to B.

The perpendicular lever arm  $r_{\perp}$  is the shortest distance from the pivot point to the line along which  $\mathbf{F}$  acts; it is shown as a dashed line in [\[link\]](#) and [\[link\]](#). Note that the line segment that defines the distance  $r_{\perp}$  is perpendicular to  $\mathbf{F}$ , as its name implies. It is sometimes easier to find or

visualize  $r_{\perp}$  than to find both  $r$  and  $\theta$ . In such cases, it may be more convenient to use  $\tau = r_{\perp}F$  rather than  $\tau = rF \sin \theta$  for torque, but both are equally valid.

The **SI unit of torque** is newtons times meters, usually written as  $\text{N} \cdot \text{m}$ . For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of  $32 \text{ N} \cdot \text{m}$  ( $0.800 \text{ m} \times 40 \text{ N} \times \sin 90^\circ$ ) relative to the hinges. If you reduce the force to 20 N, the torque is reduced to  $16 \text{ N} \cdot \text{m}$ , and so on.

The torque is always calculated with reference to some chosen pivot point. For the same applied force, a different choice for the location of the pivot will give you a different value for the torque, since both  $r$  and  $\theta$  depend on the location of the pivot. Any point in any object can be chosen to calculate the torque about that point. The object may not actually pivot about the chosen “pivot point.”

Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point, as illustrated for points B and A, respectively, in [\[link\]](#). If the object can rotate about point A, it will rotate counterclockwise, which means that the torque for the force is shown as counterclockwise relative to A. But if the object can rotate about point B, it will rotate clockwise, which means the torque for the force shown is clockwise relative to B. Also, the magnitude of the torque is greater when the lever arm is longer.

Now, *the second condition necessary to achieve equilibrium* is that *the net external torque on a system must be zero*. An external torque is one that is created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as

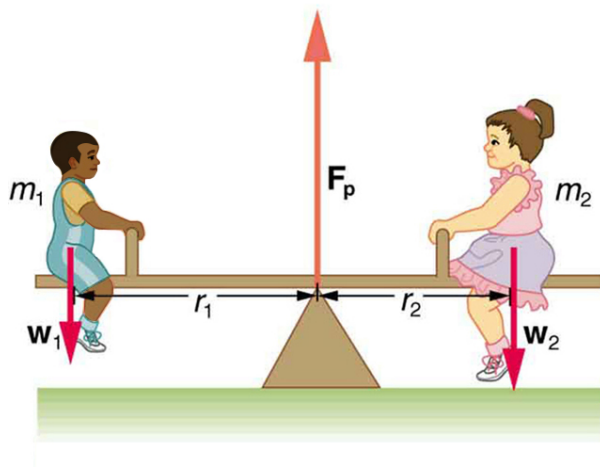
**Equation:**



$$\text{net } \tau = 0$$

where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

When two children balance a seesaw as shown in [\[link\]](#), they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely.



Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in magnitude to that of the heavier child.

**Example:****She Saw Torques On A Seesaw**

The two children shown in [\[link\]](#) are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple—more

involved examples will follow.) The first child has a mass of 26.0 kg and sits 1.60 m from the pivot.(a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is  $F_p$ , the supporting force exerted by the pivot?

### Strategy

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

### Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

### Equation:

$$\tau = rF \sin \theta.$$

Here  $\theta = 90^\circ$ , so that  $\sin \theta = 1$  for all three forces. That means  $r_\perp = r$  for all three. The torques exerted by the three forces are first,

### Equation:

$$\tau_1 = r_1 w_1$$

second,

### Equation:

$$\tau_2 = -r_2 w_2$$

and third,

### Equation:

$$\begin{aligned}\tau_p &= r_p F_p \\ &= 0 \cdot F_p \\ &= 0.\end{aligned}$$

Note that a minus sign has been inserted into the second equation because this torque is clockwise and is therefore negative by convention. Since  $F_p$  acts directly on the pivot point, the distance  $r_p$  is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

**Equation:**

$$\tau_2 = -\tau_1,$$

or

**Equation:**

$$r_2 w_2 = r_1 w_1.$$

Weight is mass times the acceleration due to gravity. Entering  $mg$  for  $w$ , we get

**Equation:**

$$r_2 m_2 g = r_1 m_1 g.$$

Solve this for the unknown  $r_2$ :

**Equation:**

$$r_2 = r_1 \frac{m_1}{m_2}.$$

The quantities on the right side of the equation are known; thus,  $r_2$  is

**Equation:**

$$r_2 = (1.60 \text{ m}) \frac{26.0 \text{ kg}}{32.0 \text{ kg}} = 1.30 \text{ m}.$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

**Solution (b)**

This part asks for a force  $F_p$ . The easiest way to find it is to use the first condition for equilibrium, which is

**Equation:**

$$\text{net } \mathbf{F} = 0.$$

The forces are all vertical, so that we are dealing with a one-dimensional problem along the vertical axis; hence, the condition can be written as

**Equation:**

$$\text{net } F_y = 0$$

where we again call the vertical axis the  $y$ -axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

**Equation:**

$$F_p - w_1 - w_2 = 0.$$

This equation yields what might have been guessed at the beginning:

**Equation:**

$$F_p = w_1 + w_2.$$

So, the pivot supplies a supporting force equal to the total weight of the system:

**Equation:**

$$F_p = m_1 g + m_2 g.$$

Entering known values gives

**Equation:**

$$\begin{aligned} F_p &= (26.0 \text{ kg})(9.80 \text{ m/s}^2) + (32.0 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 568 \text{ N.} \end{aligned}$$

### Discussion

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw's actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since  $F_p$  is exerted on the pivot point, its lever arm is zero. Hence, the torque exerted by the supporting force  $F_p$  is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. *This will not always be the case.* Always enter the correct forces—do not jump ahead to enter some ratio of masses.

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each force were exerted at a single point. This is not an approximation—the distances  $r_1$  and  $r_2$  are the distances to points directly below the **center of gravity** of each child. As we shall see in the next section, the mass and weight of a system can act as if they are located at a single point.

Finally, note that the concept of torque has an importance beyond static equilibrium. *Torque plays the same role in rotational motion that force plays in linear motion.* We will examine this in the next chapter.

#### **Note:**

##### **Take-Home Experiment**

Take a piece of modeling clay and put it on a table, then mash a cylinder down into it so that a ruler can balance on the round side of the cylinder while everything remains still. Put a penny 8 cm away from the pivot. Where would you need to put two pennies to balance? Three pennies?

## **Section Summary**

- The second condition assures those torques are also balanced. Torque is the rotational equivalent of a force in producing a rotation and is

defined to be

**Equation:**

$$\tau = rF \sin \theta$$

where  $\tau$  is torque,  $r$  is the distance from the pivot point to the point where the force is applied,  $F$  is the magnitude of the force, and  $\theta$  is the angle between  $\mathbf{F}$  and the vector directed from the point where the force acts to the pivot point. The perpendicular lever arm  $r_{\perp}$  is defined to be

**Equation:**

$$r_{\perp} = r \sin \theta$$

so that

**Equation:**

$$\tau = r_{\perp} F.$$

- The perpendicular lever arm  $r_{\perp}$  is the shortest distance from the pivot point to the line along which  $F$  acts. The SI unit for torque is newton-meter (N·m). The second condition necessary to achieve equilibrium is that the net external torque on a system must be zero:

**Equation:**

$$\text{net } \tau = 0$$

By convention, counterclockwise torques are positive, and clockwise torques are negative.

## Conceptual Questions

**Exercise:**

**Problem:**

What three factors affect the torque created by a force relative to a specific pivot point?

**Exercise:****Problem:**

A wrecking ball is being used to knock down a building. One tall unsupported concrete wall remains standing. If the wrecking ball hits the wall near the top, is the wall more likely to fall over by rotating at its base or by falling straight down? Explain your answer. How is it most likely to fall if it is struck with the same force at its base? Note that this depends on how firmly the wall is attached at its base.

**Exercise:****Problem:**

Mechanics sometimes put a length of pipe over the handle of a wrench when trying to remove a very tight bolt. How does this help? (It is also hazardous since it can break the bolt.)

**Problems & Exercises****Exercise:****Problem:**

(a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges?

---

**Solution:**

a) 46.8 N·m

b) It does not matter at what height you push. The torque depends on only the magnitude of the force applied and the perpendicular distance of the force's application from the hinges. (Children don't have a tougher time opening a door because they push lower than adults, they have a tougher time because they don't push far enough from the hinges.)

**Exercise:**

**Problem:**

When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. (a) How much torque are you exerting in newton  $\times$  meters (relative to the center of the bolt)? (b) Convert this torque to footpounds.

**Exercise:**

**Problem:**

Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.

---

**Solution:**

23.3 N

**Exercise:**

**Problem:**

Use the second condition for equilibrium (net  $\tau = 0$ ) to calculate  $F_p$  in [\[link\]](#), employing any data given or solved for in part (a) of the example.

**Exercise:**



**Problem:**

Repeat the seesaw problem in [\[link\]](#) with the center of mass of the seesaw 0.160 m to the left of the pivot (on the side of the lighter child) and assuming a mass of 12.0 kg for the seesaw. The other data given in the example remain unchanged. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium.

---

**Solution:**

Given:

**Equation:**

$$\begin{aligned}m_1 &= 26.0 \text{ kg}, m_2 = 32.0 \text{ kg}, m_s = 12.0 \text{ kg}, \\r_1 &= 1.60 \text{ m}, r_s = 0.160 \text{ m}, \text{ find (a) } r_2, \text{ (b) } F_p\end{aligned}$$

a) Since children are balancing:

**Equation:**

$$\begin{aligned}\text{net } \tau_{\text{cw}} &= -\text{net } \tau_{\text{ccw}} \\ \Rightarrow w_1 r_1 + m_s g r_s &= w_2 r_2\end{aligned}$$

So, solving for  $r_2$  gives:

**Equation:**

$$\begin{aligned}r_2 &= \frac{w_1 r_1 + m_s g r_s}{w_2} = \frac{m_1 g r_1 + m_s g r_s}{m_2 g} = \frac{m_1 r_1 + m_s r_s}{m_2} \\ &= \frac{(26.0 \text{ kg})(1.60 \text{ m}) + (12.0 \text{ kg})(0.160 \text{ m})}{32.0 \text{ kg}} \\ &= 1.36 \text{ m}\end{aligned}$$

b) Since the children are not moving:

**Equation:**

$$\text{net } F = 0 = F_p - w_1 - w_2 - w_s$$

$$\Rightarrow F_p = w_1 + w_2 + w_s$$

So that

**Equation:**

$$F_p = (26.0 \text{ kg} + 32.0 \text{ kg} + 12.0 \text{ kg})(9.80 \text{ m/s}^2)$$

$$= 686 \text{ N}$$

## Glossary

torque

turning or twisting effectiveness of a force

perpendicular lever arm

the shortest distance from the pivot point to the line along which **F** lies

SI units of torque

newton times meters, usually written as N·m

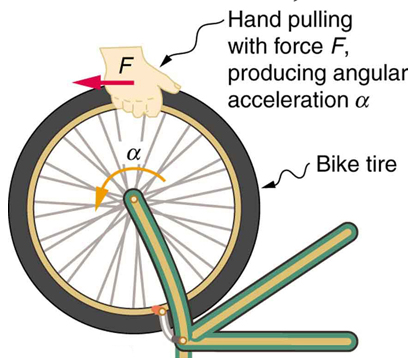
center of gravity

the point where the total weight of the body is assumed to be concentrated

## Dynamics of Rotational Motion: Rotational Inertia

- Understand the relationship between force, mass and acceleration.
- Study the turning effect of force.
- Study the analogy between force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.

If you have ever spun a bike wheel or pushed a merry-go-round, you know that force is needed to change angular velocity as seen in [\[link\]](#). In fact, your intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton's second law of motion. There are, in fact, precise rotational analogs to both force and mass.



Force is required to spin the bike wheel.

The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular acceleration. If you

push on a spoke  
closer to the axle,  
the angular  
acceleration will be  
smaller.

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force  $F$  on a point mass  $m$  that is at a distance  $r$  from a pivot point, as shown in [\[link\]](#). Because the force is perpendicular to  $r$ , an acceleration  $a = \frac{F}{m}$  is obtained in the direction of  $F$ . We can rearrange this equation such that  $F = ma$  and then look for ways to relate this expression to expressions for rotational quantities. We note that  $a = r\alpha$ , and we substitute this expression into  $F = ma$ , yielding

**Equation:**

$$F = mr\alpha.$$

Recall that **torque** is the turning effectiveness of a force. In this case, because  $\mathbf{F}$  is perpendicular to  $r$ , torque is simply  $\tau = Fr$ . So, if we multiply both sides of the equation above by  $r$ , we get torque on the left-hand side. That is,

**Equation:**

$$rF = mr^2\alpha$$

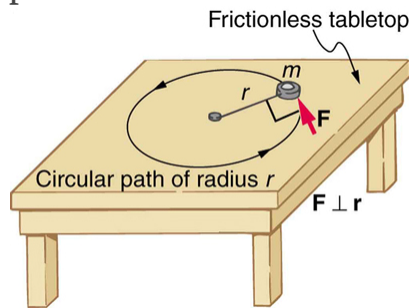
or

**Equation:**

$$\tau = mr^2\alpha.$$

This last equation is the rotational analog of Newton's second law ( $F = ma$ ), where torque is analogous to force, angular acceleration is analogous to

translational acceleration, and  $mr^2$  is analogous to mass (or inertia). The quantity  $mr^2$  is called the **rotational inertia** or **moment of inertia** of a point mass  $m$  a distance  $r$  from the center of rotation.



An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force  $F$  is applied to the object perpendicular to the radius  $r$ , causing it to accelerate about the pivot point. The force is kept perpendicular to  $r$ .

**Note:**

**Making Connections: Rotational Motion Dynamics**

Dynamics for rotational motion is completely analogous to linear or translational dynamics. Dynamics is concerned with force and mass and their effects on motion. For rotational motion, we will find direct analogs

to force and mass that behave just as we would expect from our earlier experiences.

## Rotational Inertia and Moment of Inertia

Before we can consider the rotation of anything other than a point mass like the one in [\[link\]](#), we must extend the idea of rotational inertia to all types of objects. To expand our concept of rotational inertia, we define the **moment of inertia**  $I$  of an object to be the sum of  $mr^2$  for all the point masses of which it is composed. That is,  $I = \sum mr^2$ . Here  $I$  is analogous to  $m$  in translational motion. Because of the distance  $r$ , the moment of inertia for any object depends on the chosen axis. Actually, calculating  $I$  is beyond the scope of this text except for one simple case—that of a hoop, which has all its mass at the same distance from its axis. A hoop's moment of inertia around its axis is therefore  $MR^2$ , where  $M$  is its total mass and  $R$  its radius. (We use  $M$  and  $R$  for an entire object to distinguish them from  $m$  and  $r$  for point masses.) In all other cases, we must consult [\[link\]](#) (note that the table is piece of artwork that has shapes as well as formulae) for formulas for  $I$  that have been derived from integration over the continuous body. Note that  $I$  has units of mass multiplied by distance squared ( $\text{kg} \cdot \text{m}^2$ ), as we might expect from its definition.

The general relationship among torque, moment of inertia, and angular acceleration is

**Equation:**

$$\text{net } \tau = I\alpha$$

or

**Equation:**

$$\alpha = \frac{\text{net } \tau}{I},$$

where net  $\tau$  is the total torque from all forces relative to a chosen axis. For simplicity, we will only consider torques exerted by forces in the plane of the rotation. Such torques are either positive or negative and add like ordinary numbers. The relationship in  $\tau = I\alpha$ ,  $\alpha = \frac{\text{net } \tau}{I}$  is the rotational analog to Newton's second law and is very generally applicable. This equation is actually valid for *any* torque, applied to *any* object, relative to *any* axis.

As we might expect, the larger the torque is, the larger the angular acceleration is. For example, the harder a child pushes on a merry-go-round, the faster it accelerates. Furthermore, the more massive a merry-go-round, the slower it accelerates for the same torque. The basic relationship between moment of inertia and angular acceleration is that the larger the moment of inertia, the smaller is the angular acceleration. But there is an additional twist. The moment of inertia depends not only on the mass of an object, but also on its *distribution* of mass relative to the axis around which it rotates. For example, it will be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge. The mass is the same in both cases; but the moment of inertia is much larger when the children are at the edge.

**Note:****Take-Home Experiment**

Cut out a circle that has about a 10 cm radius from stiff cardboard. Near the edge of the circle, write numbers 1 to 12 like hours on a clock face. Position the circle so that it can rotate freely about a horizontal axis through its center, like a wheel. (You could loosely nail the circle to a wall.) Hold the circle stationary and with the number 12 positioned at the top, attach a lump of blue putty (sticky material used for fixing posters to walls) at the number 3. How large does the lump need to be to just rotate the circle? Describe how you can change the moment of inertia of the circle. How does this change affect the amount of blue putty needed at the number 3 to just rotate the circle? Change the circle's moment of inertia and then try rotating the circle by using different amounts of blue putty. Repeat this process several times.

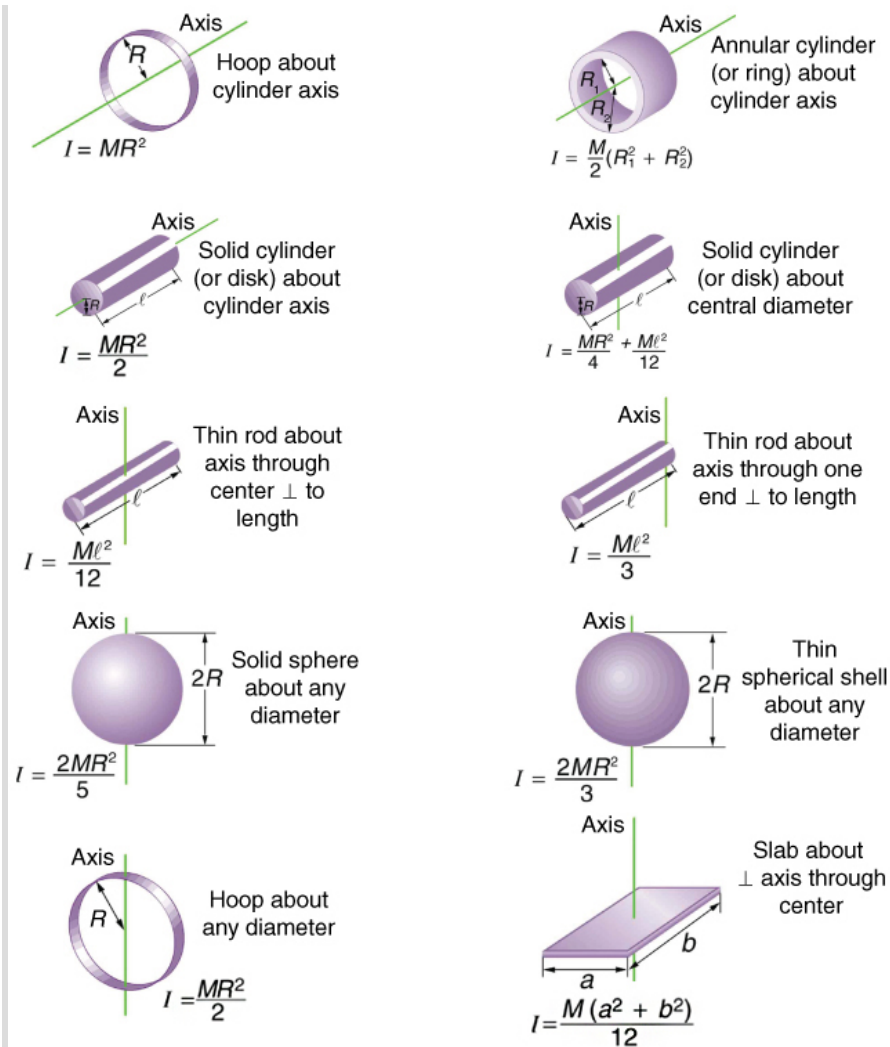
**Note:****Problem-Solving Strategy for Rotational Dynamics**

1. *Examine the situation to determine that torque and mass are involved in the rotation.* Draw a careful sketch of the situation.
2. *Determine the system of interest.*
3. *Draw a free body diagram.* That is, draw and label all external forces acting on the system of interest.
4. *Apply  $\text{net } \tau = I\alpha$ ,  $\alpha = \frac{\text{net } \tau}{I}$ , the rotational equivalent of Newton's second law, to solve the problem.* Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
5. *As always, check the solution to see if it is reasonable.*

**Note:****Making Connections**

In statics, the net torque is zero, and there is no angular acceleration. In rotational motion, net torque is the cause of angular acceleration, exactly as in Newton's second law of motion for rotation.





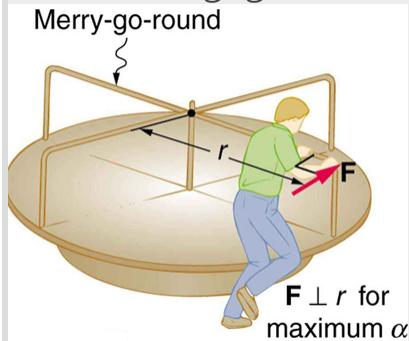
Some rotational inertias.

### Example:

#### Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in [\[link\]](#). He exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m

away from the center. Consider the merry-go-round itself to be a uniform disk with negligible retarding friction.



A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

### Strategy

Angular acceleration is given directly by the expression  $\alpha = \frac{\text{net } \tau}{I}$  :

**Equation:**

$$\alpha = \frac{\tau}{I}.$$

To solve for  $\alpha$ , we must first calculate the torque  $\tau$  (which is the same in both cases) and moment of inertia  $I$  (which is greater in the second case). To find the torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

**Equation:**

$$\tau = rF \sin \theta = (1.50 \text{ m})(250 \text{ N}) = 375 \text{ N} \cdot \text{m}.$$

### Solution for (a)

The moment of inertia of a solid disk about this axis is given in [\[link\]](#) to be

**Equation:**

$$\frac{1}{2}MR^2,$$

where  $M = 50.0 \text{ kg}$  and  $R = 1.50 \text{ m}$ , so that

**Equation:**

$$I = (0.500)(50.0 \text{ kg})(1.50 \text{ m})^2 = 56.25 \text{ kg} \cdot \text{m}^2.$$

Now, after we substitute the known values, we find the angular acceleration to be

**Equation:**

$$\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{56.25 \text{ kg} \cdot \text{m}^2} = 6.67 \frac{\text{rad}}{\text{s}^2}.$$

**Solution for (b)**

We expect the angular acceleration for the system to be less in this part, because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia  $I$ , we first find the child's moment of inertia  $I_c$  by considering the child to be equivalent to a point mass at a distance of  $1.25 \text{ m}$  from the axis. Then,

**Equation:**

$$I_c = MR^2 = (18.0 \text{ kg})(1.25 \text{ m})^2 = 28.13 \text{ kg} \cdot \text{m}^2.$$

The total moment of inertia is the sum of moments of inertia of the merry-go-round and the child (about the same axis). To justify this sum to yourself, examine the definition of  $I$ :

**Equation:**

$$I = 28.13 \text{ kg} \cdot \text{m}^2 + 56.25 \text{ kg} \cdot \text{m}^2 = 84.38 \text{ kg} \cdot \text{m}^2.$$

Substituting known values into the equation for  $\alpha$  gives

**Equation:**

$$\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{84.38 \text{ kg} \cdot \text{m}^2} = 4.44 \frac{\text{rad}}{\text{s}^2}.$$

### Discussion

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case. Summer Olympics, here he comes! Confirmation of these numbers is left as an exercise for the reader.

### Exercise:

#### Check Your Understanding

##### Problem:

Torque is the analog of force and moment of inertia is the analog of mass. Force and mass are physical quantities that depend on only one factor. For example, mass is related solely to the numbers of atoms of various types in an object. Are torque and moment of inertia similarly simple?

---

##### Solution:

No. Torque depends on three factors: force magnitude, force direction, and point of application. Moment of inertia depends on both mass and its distribution relative to the axis of rotation. So, while the analogies are precise, these rotational quantities depend on more factors.

### Section Summary

- The farther the force is applied from the pivot, the greater is the angular acceleration; angular acceleration is inversely proportional to mass.

- If we exert a force  $F$  on a point mass  $m$  that is at a distance  $r$  from a pivot point and because the force is perpendicular to  $r$ , an acceleration  $a = F / m$  is obtained in the direction of  $F$ . We can rearrange this equation such that

**Equation:**

$$F = ma,$$

and then look for ways to relate this expression to expressions for rotational quantities. We note that  $a = r\alpha$ , and we substitute this expression into  $F=ma$ , yielding

**Equation:**

$$F=mr\alpha$$

- Torque is the turning effectiveness of a force. In this case, because  $F$  is perpendicular to  $r$ , torque is simply  $\tau = rF$ . If we multiply both sides of the equation above by  $r$ , we get torque on the left-hand side. That is,

**Equation:**

$$rF = mr^2\alpha$$

or

**Equation:**

$$\tau = mr^2\alpha.$$

- The moment of inertia  $I$  of an object is the sum of  $mr^2$  for all the point masses of which it is composed. That is,

**Equation:**

$$I = \sum mr^2.$$

- The general relationship among torque, moment of inertia, and angular acceleration is

**Equation:**

$$\tau = I\alpha$$

or

**Equation:**

$$\alpha = \frac{\text{net } \tau}{I} .$$

## Conceptual Questions

**Exercise:**

**Problem:**

The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is  $ML^2/3$ . Why is this moment of inertia greater than it would be if you spun a point mass  $M$  at the location of the center of mass of the rod (at  $L/2$ )? (That would be  $ML^2/4$ .)

**Exercise:**

**Problem:**

Why is the moment of inertia of a hoop that has a mass  $M$  and a radius  $R$  greater than the moment of inertia of a disk that has the same mass and radius? Why is the moment of inertia of a spherical shell that has a mass  $M$  and a radius  $R$  greater than that of a solid sphere that has the same mass and radius?

**Exercise:**

**Problem:**

Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

**Exercise:**

**Problem:**

While reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle's frame?



The image shows a side view of a racing bicycle. Can you see evidence in the design of the wheels on this racing bicycle that their moment of inertia has been purposely reduced? (credit: Jesús Rodríguez)

**Exercise:****Problem:**

A ball slides up a frictionless ramp. It is then rolled without slipping and with the same initial velocity up another frictionless ramp (with the same slope angle). In which case does it reach a greater height, and why?

**Problems & Exercises**

**Exercise:****Problem:**

This problem considers additional aspects of example [Calculating the Effect of Mass Distribution on a Merry-Go-Round](#). (a) How long does it take the father to give the merry-go-round an angular velocity of 1.50 rad/s? (b) How many revolutions must he go through to generate this velocity? (c) If he exerts a slowing force of 300 N at a radius of 1.35 m, how long would it take him to stop them?

---

**Solution:**

(a) 0.338 s

(b) 0.0403 rev

(c) 0.313 s

**Exercise:****Problem:**

Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximately a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

**Exercise:****Problem:**

The triceps muscle in the back of the upper arm extends the forearm. This muscle in a professional boxer exerts a force of  $2.00 \times 10^3$  N with an effective perpendicular lever arm of 3.00 cm, producing an angular acceleration of the forearm of  $120 \text{ rad/s}^2$ . What is the moment of inertia of the boxer's forearm?



---

**Solution:**

$$0.50 \text{ kg} \cdot \text{m}^2$$

**Exercise:****Problem:**

A soccer player extends her lower leg in a kicking motion by exerting a force with the muscle above the knee in the front of her leg. She produces an angular acceleration of  $30.00 \text{ rad/s}^2$  and her lower leg has a moment of inertia of  $0.750 \text{ kg} \cdot \text{m}^2$ . What is the force exerted by the muscle if its effective perpendicular lever arm is  $1.90 \text{ cm}$ ?

**Exercise:****Problem:**

Suppose you exert a force of  $180 \text{ N}$  tangential to a  $0.280\text{-m}$ -radius  $75.0\text{-kg}$  grindstone (a solid disk).

(a) What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of  $20.0 \text{ N}$  exerted  $1.50 \text{ cm}$  from the axis?

---

**Solution:**

(a)  $50.4 \text{ N} \cdot \text{m}$

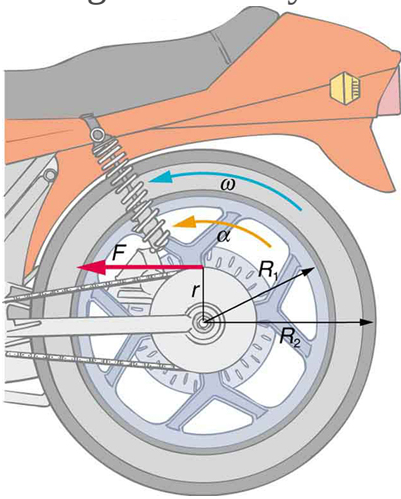
(b)  $17.1 \text{ rad/s}^2$

(c)  $17.0 \text{ rad/s}^2$

**Exercise:**

**Problem:**

Consider the 12.0 kg motorcycle wheel shown in [\[link\]](#). Assume it to be approximately an annular ring with an inner radius of 0.280 m and an outer radius of 0.330 m. The motorcycle is on its center stand, so that the wheel can spin freely. (a) If the drive chain exerts a force of 2200 N at a radius of 5.00 cm, what is the angular acceleration of the wheel? (b) What is the tangential acceleration of a point on the outer edge of the tire? (c) How long, starting from rest, does it take to reach an angular velocity of 80.0 rad/s?



A motorcycle wheel has a moment of inertia approximately that of an annular ring.

**Exercise:**

**Problem:**

Zorch, an archenemy of Superman, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Superman is not immediately concerned, because he knows Zorch can only exert a force of  $4.00 \times 10^7 \text{ N}$  (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Superman time to devote to other villains.) Explicitly show how you follow the steps found in [Problem-Solving Strategy for Rotational Dynamics](#).

---

**Solution:**

$$3.96 \times 10^{18} \text{ s}$$

$$\text{or } 1.26 \times 10^{11} \text{ y}$$

**Exercise:****Problem:**

An automobile engine can produce  $200 \text{ N} \cdot \text{m}$  of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a  $15.0 \text{ kg}$  disk that has a  $0.180 \text{ m}$  radius. The walls of each tire act like a  $2.00\text{-kg}$  annular ring that has inside radius of  $0.180 \text{ m}$  and outside radius of  $0.320 \text{ m}$ . The tread of each tire acts like a  $10.0\text{-kg}$  hoop of radius  $0.330 \text{ m}$ . The  $14.0\text{-kg}$  axle acts like a rod that has a  $2.00\text{-cm}$  radius. The  $30.0\text{-kg}$  drive shaft acts like a rod that has a  $3.20\text{-cm}$  radius.

**Exercise:**

**Problem:**

Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length ( $I = M\ell^2/3$ ), prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is  $I = M\ell^2/12$ . You will find the graphics in [\[link\]](#) useful in visualizing these rotations.

---

**Solution:**

$$I_{end} = I_{center} + m\left(\frac{l}{2}\right)^2$$

$$\text{Thus, } I_{center} = I_{end} - \frac{1}{4}ml^2 = \frac{1}{3}ml^2 - \frac{1}{4}ml^2 = \frac{1}{12}ml^2$$

**Exercise:****Problem: Unreasonable Results**

A gymnast doing a forward flip lands on the mat and exerts a  $500\text{-N} \cdot \text{m}$  torque to slow and then reverse her angular velocity. Her initial angular velocity is  $10.0\text{ rad/s}$ , and her moment of inertia is  $0.050\text{ kg} \cdot \text{m}^2$ . (a) What time is required for her to exactly reverse her spin? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

---

**Solution:**

(a)  $2.0\text{ ms}$

(b) The time interval is too short.

(c) The moment of inertia is much too small, by one to two orders of magnitude. A torque of  $500\text{ N} \cdot \text{m}$  is reasonable.

**Exercise:****Problem: Unreasonable Results**

An advertisement claims that an 800-kg car is aided by its 20.0-kg flywheel, which can accelerate the car from rest to a speed of 30.0 m/s. The flywheel is a disk with a 0.150-m radius. (a) Calculate the angular velocity the flywheel must have if 95.0% of its rotational energy is used to get the car up to speed. (b) What is unreasonable about the result? (c) Which premise is unreasonable or which premises are inconsistent?

---

**Solution:**

(a) 17,500 rpm

(b) This angular velocity is very high for a disk of this size and mass. The radial acceleration at the edge of the disk is  $> 50,000$  gs.

(c) Flywheel mass and radius should both be much greater, allowing for a lower spin rate (angular velocity).

## Glossary

torque

the turning effectiveness of a force

rotational inertia

resistance to change of rotation. The more rotational inertia an object has, the harder it is to rotate

moment of inertia

mass times the square of perpendicular distance from the rotation axis; for a point mass, it is  $I = mr^2$  and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia

## Rotational Kinetic Energy: Work and Energy Revisited

- Derive the equation for rotational work.
- Calculate rotational kinetic energy.
- Demonstrate the Law of Conservation of Energy.

In this module, we will learn about work and energy associated with rotational motion. [\[link\]](#) shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable **rotational kinetic energy**.



The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell)

Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in [Uniform Circular Motion and Gravitation](#) for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in [\[link\]](#)) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled:

**Equation:**

$$\text{net } W = (\text{net } F)\Delta s.$$

To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by  $r$ , and gather terms:

**Equation:**

$$\text{net } W = (r \text{ net } F) \frac{\Delta s}{r}.$$

We recognize that  $r \text{ net } F = \text{net } \tau$  and  $\Delta s/r = \theta$ , so that

**Equation:**

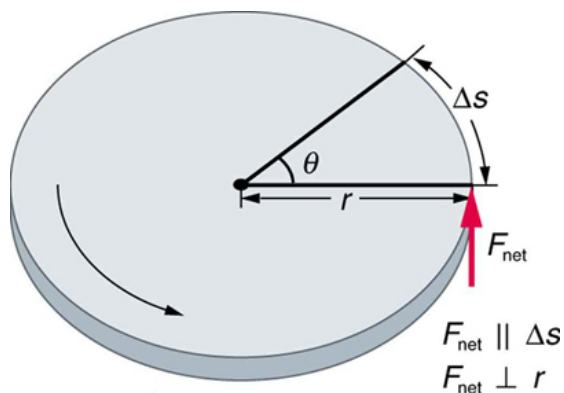
$$\text{net } W = (\text{net } \tau)\theta.$$

This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation  $\text{net } W = (\text{net } \tau)\theta$  is valid in general, even though it was derived for a special case.

To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that  $\text{net } \tau = I\alpha$ , so that

**Equation:**

$$\text{net } W = I\alpha\theta.$$



The net force on this disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus  $(\text{net } F)\Delta s$ . The net work goes into rotational kinetic energy.

**Note:**

**Making Connections**

Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in [Uniform Circular Motion and Gravitation](#).

Now, we solve one of the rotational kinematics equations for  $\alpha\theta$ . We start with the equation

**Equation:**

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$



Next, we solve for  $\alpha\theta$ :

**Equation:**

$$\alpha\theta = \frac{\omega^2 - \omega_0^2}{2}.$$

Substituting this into the equation for net  $W$  and gathering terms yields

**Equation:**

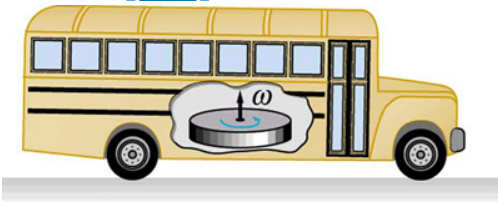
$$\text{net } W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.$$

This equation is the **work-energy theorem** for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term  $\left(\frac{1}{2}\right)I\omega^2$  to be **rotational kinetic energy**  $\text{KE}_{\text{rot}}$  for an object with a moment of inertia  $I$  and an angular velocity  $\omega$ :

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with  $I$  being analogous to  $m$  and  $\omega$  to  $v$ . Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in [\[link\]](#).



Experimental vehicles,  
such as this bus, have  
been constructed in which

rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into  $KE_{\text{rot}}$ . It can also convert translational kinetic energy, when the bus stops, into  $KE_{\text{rot}}$ . The flywheel's energy can then be used to accelerate, to go up another hill, or to keep the bus from going against friction.

**Example:****Calculating the Work and Energy for Spinning a Grindstone**

Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in [\[link\]](#). In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad (57.3°)? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)

**Strategy**

To find the work, we can use the equation  $W = (\text{net } \tau)\theta$ . We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the

kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in  $KE_{\text{rot}} = \frac{1}{2} I \omega^2$ .

**Solution for (a)**

The net work is expressed in the equation

**Equation:**

$$\text{net } W = (\text{net } \tau)\theta,$$

where net  $\tau$  is the applied force multiplied by the radius ( $rF$ ) because there is no retarding friction, and the force is perpendicular to  $r$ . The angle  $\theta$  is given. Substituting the given values in the equation above yields

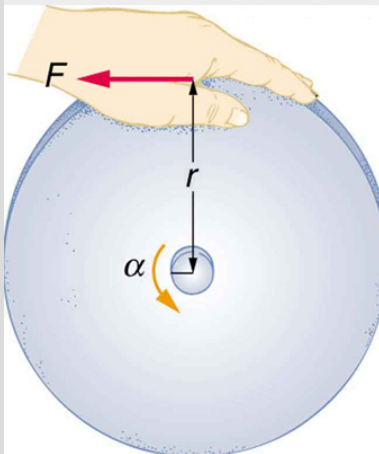
**Equation:**

$$\begin{aligned}\text{net } W &= rF\theta = (0.320 \text{ m})(200 \text{ N})(1.00 \text{ rad}) \\ &= 64.0 \text{ N} \cdot \text{m}.\end{aligned}$$

Noting that  $1 \text{ N} \cdot \text{m} = 1 \text{ J}$ ,

**Equation:**

$$\text{net } W = 64.0 \text{ J}.$$



A large grindstone is given a spin by a person grasping its outer edge.

**Solution for (b)**

To find  $\omega$  from the given information requires more than one step. We start with the kinematic relationship in the equation

**Equation:**

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Note that  $\omega_0 = 0$  because we start from rest. Taking the square root of the resulting equation gives

**Equation:**

$$\omega = (2\alpha\theta)^{1/2}.$$

Now we need to find  $\alpha$ . One possibility is

**Equation:**

$$\alpha = \frac{\text{net } \tau}{I},$$

where the torque is

**Equation:**

$$\text{net } \tau = rF = (0.320 \text{ m})(200 \text{ N}) = 64.0 \text{ N} \cdot \text{m}.$$

The formula for the moment of inertia for a disk is found in [\[link\]](#):

**Equation:**

$$I = \frac{1}{2}MR^2 = 0.5(85.0 \text{ kg})(0.320 \text{ m})^2 = 4.352 \text{ kg} \cdot \text{m}^2.$$

Substituting the values of torque and moment of inertia into the expression for  $\alpha$ , we obtain

**Equation:**

$$\alpha = \frac{64.0 \text{ N} \cdot \text{m}}{4.352 \text{ kg} \cdot \text{m}^2} = 14.7 \frac{\text{rad}}{\text{s}^2}.$$

Now, substitute this value and the given value for  $\theta$  into the above expression for  $\omega$ :

**Equation:**

$$\omega = (2\alpha\theta)^{1/2} = \left[ 2 \left( 14.7 \frac{\text{rad}}{\text{s}^2} \right) (1.00 \text{ rad}) \right]^{1/2} = 5.42 \frac{\text{rad}}{\text{s}}.$$

**Solution for (c)**

The final rotational kinetic energy is

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

Both  $I$  and  $\omega$  were found above. Thus,

**Equation:**

$$\text{KE}_{\text{rot}} = (0.5)(4.352 \text{ kg} \cdot \text{m}^2)(5.42 \text{ rad/s})^2 = 64.0 \text{ J}.$$

**Discussion**

The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples.

Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash

is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground.

**Note:**

**Problem-Solving Strategy for Rotational Energy**

1. *Determine that energy or work is involved in the rotation.*
2. *Determine the system of interest. A sketch usually helps.*
3. *Analyze the situation to determine the types of work and energy involved.*
4. *For closed systems, mechanical energy is conserved. That is,  $KE_i + PE_i = KE_f + PE_f$ . Note that  $KE_i$  and  $KE_f$  may each include translational and rotational contributions.*
5. *For open systems, mechanical energy may not be conserved, and other forms of energy (referred to previously as OE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary.*
6. *Eliminate terms wherever possible to simplify the algebra.*
7. *Check the answer to see if it is reasonable.*

**Example:**

**Calculating Helicopter Energies**

A typical small rescue helicopter, similar to the one in [\[link\]](#), has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational

energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?

### Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy.

### Solution for (a)

The rotational kinetic energy is

### Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find  $\text{KE}_{\text{rot}}$ . The angular velocity  $\omega$  is

### Equation:

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade will be that of a thin rod rotated about its end, found in [\[link\]](#). The total  $I$  is four times this moment of inertia, because there are four blades. Thus,

### Equation:

$$I = 4 \frac{M\ell^2}{3} = 4 \times \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} = 1067 \text{ kg} \cdot \text{m}^2.$$

Entering  $\omega$  and  $I$  into the expression for rotational kinetic energy gives

### Equation:

$$\begin{aligned} \text{KE}_{\text{rot}} &= 0.5(1067 \text{ kg} \cdot \text{m}^2)(31.4 \text{ rad/s})^2 \\ &= 5.26 \times 10^5 \text{ J} \end{aligned}$$

### Solution for (b)

Translational kinetic energy was defined in [Uniform Circular Motion and Gravitation](#). Entering the given values of mass and velocity, we obtain

**Equation:**

$$\text{KE}_{\text{trans}} = \frac{1}{2}mv^2 = (0.5)(1000 \text{ kg})(20.0 \text{ m/s})^2 = 2.00 \times 10^5 \text{ J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

**Equation:**

$$\frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380.$$

### **Solution for (c)**

At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies:

**Equation:**

$$\text{KE}_{\text{rot}} = \text{PE}_{\text{grav}}$$

or

**Equation:**

$$\frac{1}{2}I\omega^2 = mgh.$$

We now solve for  $h$  and substitute known values into the resulting equation

**Equation:**

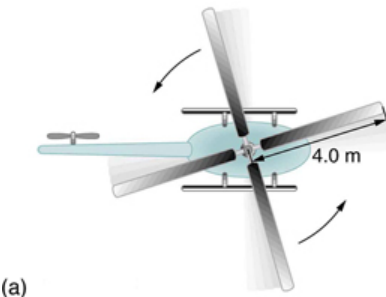
$$h = \frac{\frac{1}{2}I\omega^2}{mg} = \frac{5.26 \times 10^5 \text{ J}}{(1000 \text{ kg})(9.80 \text{ m/s}^2)} = 53.7 \text{ m}.$$

### **Discussion**

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its



spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades.



(a)



(b)

The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously

provide lift and put  
significant  
rotational energy  
into the blades.

The second image  
shows a helicopter  
from the Auckland  
Westpac Rescue  
Helicopter Service.

Over 50,000 lives  
have been saved  
since its operations  
beginning in 1973.

Here, a water  
rescue operation is  
shown. (credit: 111  
Emergency, Flickr)

**Note:**

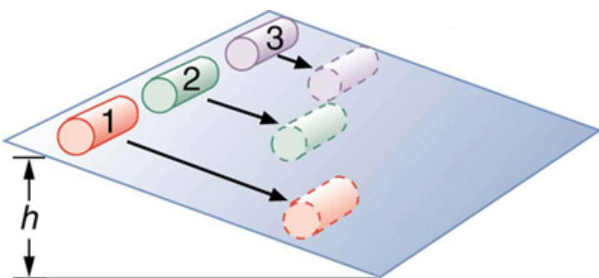
**Making Connections**

Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE . [Uniform Circular Motion and Gravitation](#) has a detailed treatment of conservation of energy.

## **How Thick Is the Soup? Or Why Don't All Objects Roll Downhill at the Same Rate?**

One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest?

The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy  $PE_{\text{grav}}$ , which is converted entirely to KE, provided each rolls without slipping. KE, however, can take the form of  $KE_{\text{trans}}$  or  $KE_{\text{rot}}$ , and total KE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in [\[link\]](#).



Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in

third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE.

Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives

**Equation:**

$$PE_i = KE_f.$$

More specifically,

**Equation:**

$$PE_{\text{grav}} = KE_{\text{trans}} + KE_{\text{rot}}$$

or

**Equation:**

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

So, the initial  $mgh$  is divided between translational kinetic energy and rotational kinetic energy; and the greater  $I$  is, the less energy goes into translation. If the can slides down without friction, then  $\omega = 0$  and all the energy goes into translation; thus, the can goes faster.

**Note:**

Take-Home Experiment

Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand.

**Example:**

**Calculating the Speed of a Cylinder Rolling Down an Incline**

Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

**Strategy**

We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with  $v$  as the only unknown.

**Solution**

Conservation of energy for this situation is written as described above:

**Equation:**

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

Before we can solve for  $v$ , we must get an expression for  $I$  from [\[link\]](#). Because  $v$  and  $\omega$  are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship  $\omega = v/R$  into the expression. These substitutions yield

**Equation:**

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v^2}{R^2}\right).$$

Interestingly, the cylinder's radius  $R$  and mass  $m$  cancel, yielding

**Equation:**

$$gh = \frac{1}{2}v^2 + \frac{1}{4}v^2 = \frac{3}{4}v^2.$$

Solving algebraically, the equation for the final velocity  $v$  gives

**Equation:**

$$v = \left( \frac{4gh}{3} \right)^{1/2}.$$

Substituting known values into the resulting expression yields

**Equation:**

$$v = \frac{4 \left( 9.80 \text{ m/s}^2 \right) (2.00 \text{ m})}{3}^{1/2} = 5.11 \text{ m/s}.$$

### Discussion

Because  $m$  and  $R$  cancel, the result  $v = \left( \frac{4}{3} gh \right)^{1/2}$  is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus,  $\frac{1}{2}mv^2 = mgh$  and  $v = (2gh)^{1/2}$ , which is 22% greater than  $(4gh/3)^{1/2}$ . That is, the cylinder would go faster at the bottom.

### Exercise:

#### Check Your Understanding

##### Problem:

##### Analogy of Rotational and Translational Kinetic Energy

Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy.

---

**Solution:**

Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth.

**Note:**

PhET Explorations: My Solar System

Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other.

[https://phet.colorado.edu/sims/my-solar-system/my-solar-system\\_en.html](https://phet.colorado.edu/sims/my-solar-system/my-solar-system_en.html)

**Section Summary**

- The rotational kinetic energy  $\text{KE}_{\text{rot}}$  for an object with a moment of inertia  $I$  and an angular velocity  $\omega$  is given by

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

- Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and

maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades.

- Work and energy in rotational motion are completely analogous to work and energy in translational motion.
- The equation for the **work-energy theorem** for rotational motion is,  
**Equation:**

$$\text{net } W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.$$

## Conceptual Questions

### Exercise:

#### Problem:

Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user's hand.

### Exercise:

#### Problem:

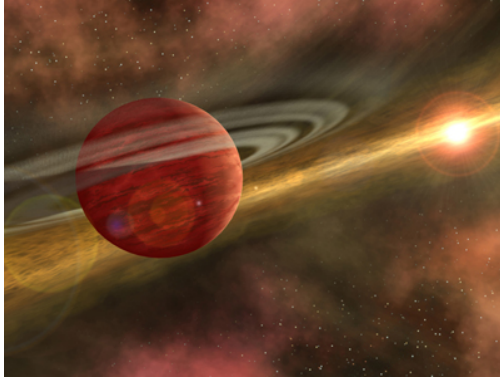
What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step.

### Exercise:

#### Problem:

The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from?





An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA)

## Problems & Exercises

### Exercise:

#### Problem:

This problem considers energy and work aspects of [\[link\]](#)—use data from that example as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they have an angular velocity of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutions

---

#### Solution:

(a) 185 J

(b) 0.0785 rev

(c)  $W = 9.81 \text{ N}$

**Exercise:**

**Problem:**

What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest?

**Exercise:**

**Problem:**

(a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

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**Solution:**

(a)  $2.57 \times 10^{29} \text{ J}$

(b)  $\text{KE}_{\text{rot}} = 2.65 \times 10^{33} \text{ J}$

**Exercise:**

**Problem:**

Calculate the rotational kinetic energy in the motorcycle wheel ([\[link\]](#)) if its angular velocity is 120 rad/s. Assume  $M = 12.0 \text{ kg}$ ,  $R_1 = 0.280 \text{ m}$ , and  $R_2 = 0.330 \text{ m}$ .

**Exercise:**

**Problem:**

A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is  $0.500 \text{ kg} \cdot \text{m}^2$ , what is the rotational kinetic energy of the forearm?

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**Solution:****Equation:**

$$KE_{\text{rot}} = 434 \text{ J}$$

**Exercise:****Problem:**

While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is  $3.75 \text{ kg} \cdot \text{m}^2$  and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance).

**Exercise:****Problem:**

A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10,000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the top of the hill? Explicitly show how you follow the steps in the [Problem-Solving Strategy for Rotational Energy](#).

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**Solution:**

(a)  $128 \text{ rad/s}$

(b)  $19.9 \text{ m}$

**Exercise:**

**Problem:**

A ball with an initial velocity of  $8.00 \text{ m/s}$  rolls up a hill without slipping. Treating the ball as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

**Exercise:**

**Problem:**

While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is  $10.0 \text{ kg}$  at a distance of  $28.0 \text{ cm}$  from the knee joint, the moment of inertia of the lower leg is  $0.900 \text{ kg} \cdot \text{m}^2$ , the muscle force is  $1500 \text{ N}$ , and its effective perpendicular lever arm is  $3.00 \text{ cm}$ . (b) How much work is done if the leg rotates through an angle of  $20.0^\circ$  with a constant force exerted by the muscle?

---

**Solution:**

(a)  $10.4 \text{ rad/s}^2$

(b)  $\text{net } W = 6.11 \text{ J}$

**Exercise:**

**Problem:**

To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of  $60.0^\circ$ . (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of  $0.250 \text{ kg} \cdot \text{m}^2$ , and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do?

**Exercise:****Problem:**

Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion.

**Exercise:****Problem:**

What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of  $MR^2$ , where  $M$  is the mass of the object and  $R$  is its radius.

**Exercise:**

**Problem:**

Suppose a 200-kg motorcycle has two wheels like, [the one described in Problem 10.15](#) and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest?

**Exercise:****Problem:**

In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher's arm given its moment of inertia is  $0.720 \text{ kg} \cdot \text{m}^2$  and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg?

---

**Solution:**

(a) 1.49 kJ

(b)  $2.52 \times 10^4 \text{ N}$

**Exercise:****Problem: Construct Your Own Problem**

Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a “force multiplied by distance” calculation and compare it to the skater's increase in kinetic energy.

**Glossary**

work-energy theorem

if one or more external forces act upon a rigid object, causing its kinetic energy to change from  $KE_1$  to  $KE_2$ , then the work  $W$  done by the net force is equal to the change in kinetic energy

rotational kinetic energy

the kinetic energy due to the rotation of an object. This is part of its total kinetic energy

## Angular Momentum and Its Conservation

- Understand the analogy between angular momentum and linear momentum.
- Observe the relationship between torque and angular momentum.
- Apply the law of conservation of angular momentum.

Why does Earth keep on spinning? What started it spinning to begin with? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster? Questions like these have answers based in angular momentum, the rotational analog to linear momentum.

By now the pattern is clear—every rotational phenomenon has a direct translational analog. It seems quite reasonable, then, to define **angular momentum**  $L$  as

**Equation:**

$$L = I\omega.$$

This equation is an analog to the definition of linear momentum as  $p = mv$ . Units for linear momentum are  $\text{kg} \cdot \text{m/s}$  while units for angular momentum are  $\text{kg} \cdot \text{m}^2/\text{s}$ . As we would expect, an object that has a large moment of inertia  $I$ , such as Earth, has a very large angular momentum. An object that has a large angular velocity  $\omega$ , such as a centrifuge, also has a rather large angular momentum.

### **Note:**

#### **Making Connections**

Angular momentum is completely analogous to linear momentum, first presented in [Uniform Circular Motion and Gravitation](#). It has the same implications in terms of carrying rotation forward, and it is conserved when the net external torque is zero. Angular momentum, like linear momentum, is also a property of the atoms and subatomic particles.



**Example:****Calculating Angular Momentum of the Earth****Strategy**

No information is given in the statement of the problem; so we must look up pertinent data before we can calculate  $L = I\omega$ . First, according to [\[link\]](#), the formula for the moment of inertia of a sphere is

**Equation:**

$$I = \frac{2MR^2}{5}$$

so that

**Equation:**

$$L = I\omega = \frac{2MR^2\omega}{5}.$$

Earth's mass  $M$  is  $5.979 \times 10^{24}$  kg and its radius  $R$  is  $6.376 \times 10^6$  m. The Earth's angular velocity  $\omega$  is, of course, exactly one revolution per day, but we must convert  $\omega$  to radians per second to do the calculation in SI units.

**Solution**

Substituting known information into the expression for  $L$  and converting  $\omega$  to radians per second gives

**Equation:**

$$\begin{aligned} L &= 0.4(5.979 \times 10^{24} \text{ kg})(6.376 \times 10^6 \text{ m})^2 \left(\frac{1 \text{ rev}}{\text{d}}\right) \\ &= 9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2 \cdot \text{rev/d}. \end{aligned}$$

Substituting  $2\pi$  rad for 1 rev and  $8.64 \times 10^4$  s for 1 day gives

**Equation:**

$$\begin{aligned} L &= (9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2) \left(\frac{2\pi \text{ rad/rev}}{8.64 \times 10^4 \text{ s/d}}\right) (1 \text{ rev/d}) \\ &= 7.07 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}. \end{aligned}$$

**Discussion**

This number is large, demonstrating that Earth, as expected, has a tremendous angular momentum. The answer is approximate, because we have assumed a constant density for Earth in order to estimate its moment of inertia.

When you push a merry-go-round, spin a bike wheel, or open a door, you exert a torque. If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases. The greater the net torque, the more rapid the increase in  $L$ . The relationship between torque and angular momentum is

**Equation:**

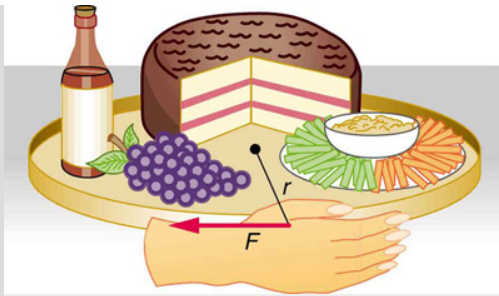
$$\text{net } \tau = \frac{\Delta L}{\Delta t}.$$

This expression is exactly analogous to the relationship between force and linear momentum,  $F = \Delta p / \Delta t$ . The equation  $\text{net } \tau = \frac{\Delta L}{\Delta t}$  is very fundamental and broadly applicable. It is, in fact, the rotational form of Newton's second law.

**Example:**

**Calculating the Torque Putting Angular Momentum Into a Lazy Susan**

[\[link\]](#) shows a Lazy Susan food tray being rotated by a person in quest of sustenance. Suppose the person exerts a 2.50 N force perpendicular to the lazy Susan's 0.260-m radius for 0.150 s. (a) What is the final angular momentum of the lazy Susan if it starts from rest, assuming friction is negligible? (b) What is the final angular velocity of the lazy Susan, given that its mass is 4.00 kg and assuming its moment of inertia is that of a disk?



A partygoer exerts a torque on a lazy Susan to make it rotate. The equation  $\text{net } \tau = \frac{\Delta L}{\Delta t}$  gives the relationship between torque and the angular momentum produced.

### Strategy

We can find the angular momentum by solving  $\text{net } \tau = \frac{\Delta L}{\Delta t}$  for  $\Delta L$ , and using the given information to calculate the torque. The final angular momentum equals the change in angular momentum, because the lazy Susan starts from rest. That is,  $\Delta L = L$ . To find the final velocity, we must calculate  $\omega$  from the definition of  $L$  in  $L = I\omega$ .

### Solution for (a)

Solving  $\text{net } \tau = \frac{\Delta L}{\Delta t}$  for  $\Delta L$  gives

**Equation:**

$$\Delta L = (\text{net } \tau)\Delta t.$$

Because the force is perpendicular to  $r$ , we see that  $\text{net } \tau = rF$ , so that

**Equation:**

$$\begin{aligned} L &= rF\Delta t = (0.260 \text{ m})(2.50 \text{ N})(0.150 \text{ s}) \\ &= 9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}. \end{aligned}$$

### Solution for (b)

The final angular velocity can be calculated from the definition of angular momentum,

**Equation:**

$$L = I\omega.$$

Solving for  $\omega$  and substituting the formula for the moment of inertia of a disk into the resulting equation gives

**Equation:**

$$\omega = \frac{L}{I} = \frac{L}{\frac{1}{2}MR^2}.$$

And substituting known values into the preceding equation yields

**Equation:**

$$\omega = \frac{9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}}{(0.500)(4.00 \text{ kg})(0.260 \text{ m})} = 0.721 \text{ rad/s}.$$

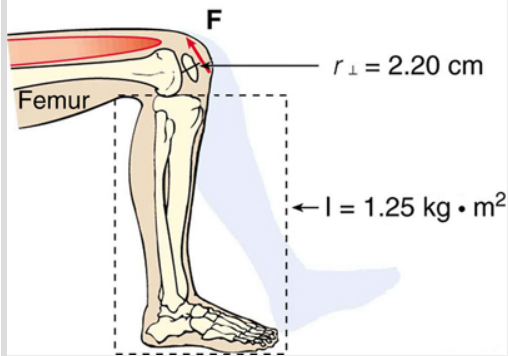
### Discussion

Note that the imparted angular momentum does not depend on any property of the object but only on torque and time. The final angular velocity is equivalent to one revolution in 8.71 s (determination of the time period is left as an exercise for the reader), which is about right for a lazy Susan.

### Example:

#### Calculating the Torque in a Kick

The person whose leg is shown in [\[link\]](#) kicks his leg by exerting a 2000-N force with his upper leg muscle. The effective perpendicular lever arm is 2.20 cm. Given the moment of inertia of the lower leg is  $1.25 \text{ kg} \cdot \text{m}^2$ , (a) find the angular acceleration of the leg. (b) Neglecting the gravitational force, what is the rotational kinetic energy of the leg after it has rotated through  $57.3^\circ$  (1.00 rad)?



The muscle in the upper leg gives the lower leg an angular acceleration and imparts rotational kinetic energy to it by exerting a torque about the knee.  $\mathbf{F}$  is a vector that is perpendicular to  $r$ . This example examines the situation.

### Strategy

The angular acceleration can be found using the rotational analog to Newton's second law, or  $\alpha = \text{net } \tau / I$ . The moment of inertia  $I$  is given and the torque can be found easily from the given force and perpendicular lever arm. Once the angular acceleration  $\alpha$  is known, the final angular velocity and rotational kinetic energy can be calculated.

### Solution to (a)

From the rotational analog to Newton's second law, the angular acceleration  $\alpha$  is

### Equation:

$$\alpha = \frac{\text{net } \tau}{I}.$$

Because the force and the perpendicular lever arm are given and the leg is vertical so that its weight does not create a torque, the net torque is thus

**Equation:**

$$\begin{aligned}\text{net } \tau &= r_{\perp} F \\ &= (0.0220 \text{ m})(2000 \text{ N}) \\ &= 44.0 \text{ N} \cdot \text{m}.\end{aligned}$$

Substituting this value for the torque and the given value for the moment of inertia into the expression for  $\alpha$  gives

**Equation:**

$$\alpha = \frac{44.0 \text{ N} \cdot \text{m}}{1.25 \text{ kg} \cdot \text{m}^2} = 35.2 \text{ rad/s}^2.$$

**Solution to (b)**

The final angular velocity can be calculated from the kinematic expression

**Equation:**

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

or

**Equation:**

$$\omega^2 = 2\alpha\theta$$

because the initial angular velocity is zero. The kinetic energy of rotation is

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2$$

so it is most convenient to use the value of  $\omega^2$  just found and the given value for the moment of inertia. The kinetic energy is then

**Equation:**

$$\begin{aligned}\text{KE}_{\text{rot}} &= 0.5(1.25 \text{ kg} \cdot \text{m}^2)(70.4 \text{ rad}^2/\text{s}^2) \\ &= 44.0 \text{ J}\end{aligned}$$

**Discussion**

These values are reasonable for a person kicking his leg starting from the position shown. The weight of the leg can be neglected in part (a) because it exerts no torque when the center of gravity of the lower leg is directly beneath the pivot in the knee. In part (b), the force exerted by the upper leg is so large that its torque is much greater than that created by the weight of the lower leg as it rotates. The rotational kinetic energy given to the lower leg is enough that it could give a ball a significant velocity by transferring some of this energy in a kick.

**Note:**

**Making Connections: Conservation Laws**

Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

## Conservation of Angular Momentum

We can now understand why Earth keeps on spinning. As we saw in the previous example,  $\Delta L = (\text{net } \tau)\Delta t$ . This equation means that, to change angular momentum, a torque must act over some period of time. Because Earth has a large angular momentum, a large torque acting over a long time is needed to change its rate of spin. So what external torques are there? Tidal friction exerts torque that is slowing Earth's rotation, but tens of millions of years must pass before the change is very significant. Recent research indicates the length of the day was 18 h some 900 million years ago. Only the tides exert significant retarding torques on Earth, and so it will continue to spin, although ever more slowly, for many billions of years.

What we have here is, in fact, another conservation law. If the net torque is zero, then angular momentum is constant or *conserved*. We can see this rigorously by considering net  $\tau = \frac{\Delta L}{\Delta t}$  for the situation in which the net torque is zero. In that case,

**Equation:**

$$\text{net}\tau = 0$$

implying that

**Equation:**

$$\frac{\Delta L}{\Delta t} = 0.$$

If the change in angular momentum  $\Delta L$  is zero, then the angular momentum is constant; thus,

**Equation:**

$$L = \text{constant} \ (\text{net } \tau = 0)$$

or

**Equation:**

$$L = L'(\text{net}\tau = 0).$$

These expressions are the **law of conservation of angular momentum**. Conservation laws are as scarce as they are important.

An example of conservation of angular momentum is seen in [\[link\]](#), in which an ice skater is executing a spin. The net torque on her is very close to zero, because there is relatively little friction between her skates and the ice and because the friction is exerted very close to the pivot point. (Both  $F$  and  $r$  are small, and so  $\tau$  is negligibly small.) Consequently, she can spin for quite some time. She can do something else, too. She can increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

**Equation:**

$$L = L'.$$

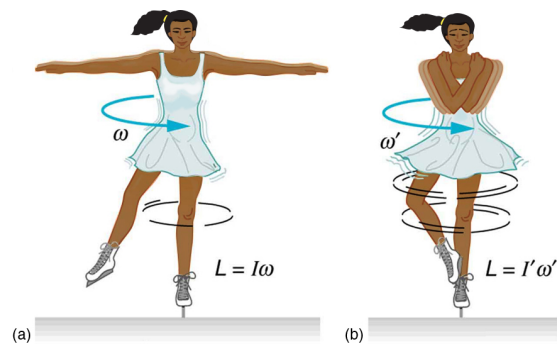


Expressing this equation in terms of the moment of inertia,

**Equation:**

$$I\omega = I'\omega',$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because  $I'$  is smaller, the angular velocity  $\omega'$  must increase to keep the angular momentum constant. The change can be dramatic, as the following example shows.



(a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

**Example:****Calculating the Angular Momentum of a Spinning Skater**

Suppose an ice skater, such as the one in [\[link\]](#), is spinning at 0.800 rev/s with her arms extended. She has a moment of inertia of  $2.34 \text{ kg} \cdot \text{m}^2$  with her arms extended and of  $0.363 \text{ kg} \cdot \text{m}^2$  with her arms close to her body. (These moments of inertia are based on reasonable assumptions about a 60.0-kg skater.) (a) What is her angular velocity in revolutions per second after she pulls in her arms? (b) What is her rotational kinetic energy before and after she does this?

**Strategy**

In the first part of the problem, we are looking for the skater's angular velocity  $\omega'$  after she has pulled in her arms. To find this quantity, we use the conservation of angular momentum and note that the moments of inertia and initial angular velocity are given. To find the initial and final kinetic energies, we use the definition of rotational kinetic energy given by

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

**Solution for (a)**

Because torque is negligible (as discussed above), the conservation of angular momentum given in  $I\omega = I'\omega'$  is applicable. Thus,

**Equation:**

$$L = L'$$

or

**Equation:**

$$I\omega = I'\omega'$$

Solving for  $\omega'$  and substituting known values into the resulting equation gives

**Equation:**

$$\begin{aligned}\omega' &= \frac{I}{I'}\omega = \left(\frac{2.34 \text{ kg}\cdot\text{m}^2}{0.363 \text{ kg}\cdot\text{m}^2}\right)(0.800 \text{ rev/s}) \\ &= 5.16 \text{ rev/s}.\end{aligned}$$

### **Solution for (b)**

Rotational kinetic energy is given by

**Equation:**

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The initial value is found by substituting known values into the equation and converting the angular velocity to rad/s:

**Equation:**

$$\begin{aligned}\text{KE}_{\text{rot}} &= (0.5)(2.34 \text{ kg}\cdot\text{m}^2)((0.800 \text{ rev/s})(2\pi \text{ rad/rev}))^2 \\ &= 29.6 \text{ J}.\end{aligned}$$

The final rotational kinetic energy is

**Equation:**

$$\text{KE}_{\text{rot}}' = \frac{1}{2}I'\omega'^2.$$

Substituting known values into this equation gives

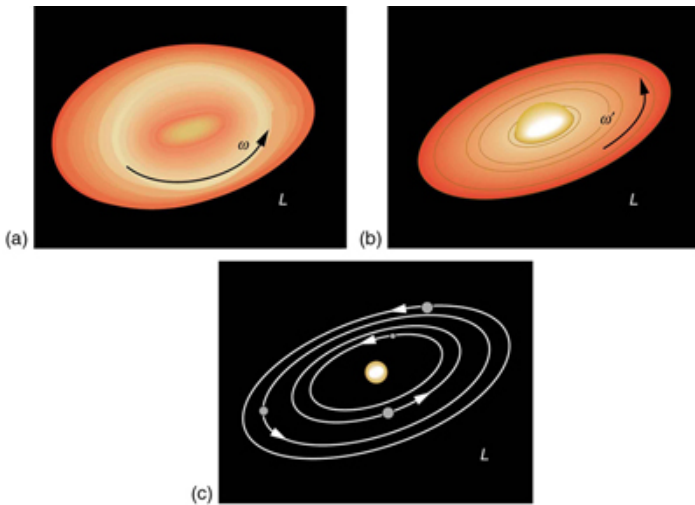
**Equation:**

$$\begin{aligned}KE_{\text{rot}}' &= (0.5)(0.363 \text{ kg}\cdot\text{m}^2)[(5.16 \text{ rev/s})(2\pi \text{ rad/rev})]^2 \\ &= 191 \text{ J}.\end{aligned}$$

### **Discussion**

In both parts, there is an impressive increase. First, the final angular velocity is large, although most world-class skaters can achieve spin rates about this great. Second, the final kinetic energy is much greater than the initial kinetic energy. The increase in rotational kinetic energy comes from work done by the skater in pulling in her arms. This work is internal work that depletes some of the skater's food energy.

There are several other examples of objects that increase their rate of spin because something reduced their moment of inertia. Tornadoes are one example. Storm systems that create tornadoes are slowly rotating. When the radius of rotation narrows, even in a local region, angular velocity increases, sometimes to the furious level of a tornado. Earth is another example. Our planet was born from a huge cloud of gas and dust, the rotation of which came from turbulence in an even larger cloud. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result. (See [\[link\]](#).)



The Solar System coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud.

In case of human motion, one would not expect angular momentum to be conserved when a body interacts with the environment as its foot pushes off the ground. Astronauts floating in space aboard the International Space Station have no angular momentum relative to the inside of the ship if they

are motionless. Their bodies will continue to have this zero value no matter how they twist about as long as they do not give themselves a push off the side of the vessel.

**Exercise:**

**Check Your Understanding**

**Problem:**

Is angular momentum completely analogous to linear momentum?  
What, if any, are their differences?

---

**Solution:**

Yes, angular and linear momentums are completely analogous. While they are exact analogs they have different units and are not directly inter-convertible like forms of energy are.

**Section Summary**

- Every rotational phenomenon has a direct translational analog , likewise angular momentum  $L$  can be defined as  $L = I\omega$ .
- This equation is an analog to the definition of linear momentum as  $p = mv$ . The relationship between torque and angular momentum is  $\tau_{\text{net}} = \frac{\Delta L}{\Delta t}$ .
- Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

**Conceptual Questions**

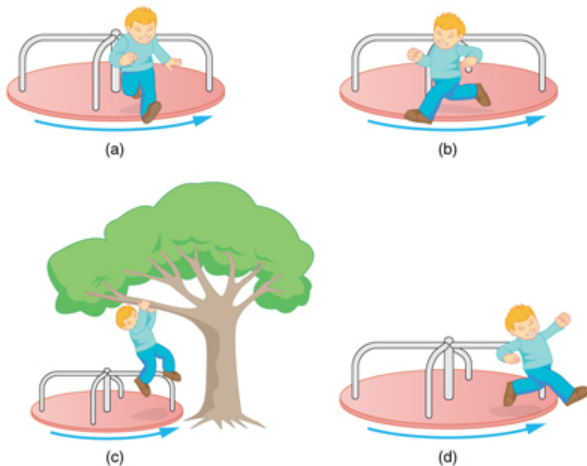
**Exercise:**

**Problem:**

When you start the engine of your car with the transmission in neutral, you notice that the car rocks in the opposite sense of the engine's rotation. Explain in terms of conservation of angular momentum. Is the angular momentum of the car conserved for long (for more than a few seconds)?

**Exercise:****Problem:**

Suppose a child walks from the outer edge of a rotating merry-go-round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer.



A child may jump off a merry-go-round in a variety of directions.

**Exercise:**

**Problem:**

Suppose a child gets off a rotating merry-go-round. Does the angular velocity of the merry-go-round increase, decrease, or remain the same if: (a) He jumps off radially? (b) He jumps backward to land motionless? (c) He jumps straight up and hangs onto an overhead tree branch? (d) He jumps off forward, tangential to the edge? Explain your answers. (Refer to [\[link\]](#)).

**Exercise:****Problem:**

Helicopters have a small propeller on their tail to keep them from rotating in the opposite direction of their main lifting blades. Explain in terms of Newton's third law why the helicopter body rotates in the opposite direction to the blades.

**Exercise:****Problem:**

Whenever a helicopter has two sets of lifting blades, they rotate in opposite directions (and there will be no tail propeller). Explain why it is best to have the blades rotate in opposite directions.

**Exercise:****Problem:**

Describe how work is done by a skater pulling in her arms during a spin. In particular, identify the force she exerts on each arm to pull it in and the distance each moves, noting that a component of the force is in the direction moved. Why is angular momentum not increased by this action?

**Exercise:**

**Problem:**

When there is a global heating trend on Earth, the atmosphere expands and the length of the day increases very slightly. Explain why the length of a day increases.

**Exercise:****Problem:**

Nearly all conventional piston engines have flywheels on them to smooth out engine vibrations caused by the thrust of individual piston firings. Why does the flywheel have this effect?

**Exercise:****Problem:**

Jet turbines spin rapidly. They are designed to fly apart if something makes them seize suddenly, rather than transfer angular momentum to the plane's wing, possibly tearing it off. Explain how flying apart conserves angular momentum without transferring it to the wing.

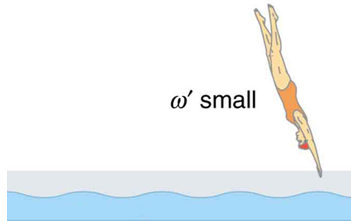
**Exercise:****Problem:**

An astronaut tightens a bolt on a satellite in orbit. He rotates in a direction opposite to that of the bolt, and the satellite rotates in the same direction as the bolt. Explain why. If a handhold is available on the satellite, can this counter-rotation be prevented? Explain your answer.

**Exercise:****Problem:**

Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down. Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momenta.





The diver spins rapidly when curled up and slows when she extends her limbs before entering the water.

**Exercise:**

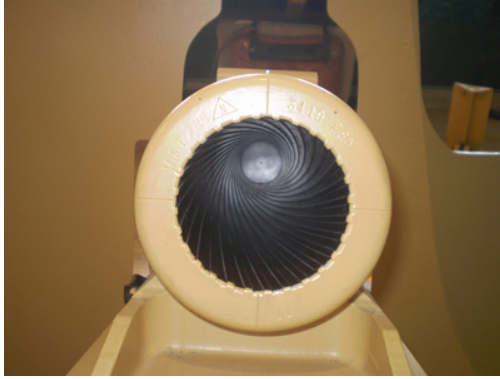
**Problem:**

Draw a free body diagram to show how a diver gains angular momentum when leaving the diving board.

**Exercise:**

**Problem:**

In terms of angular momentum, what is the advantage of giving a football or a rifle bullet a spin when throwing or releasing it?



The image shows a view down the barrel of a cannon, emphasizing its rifling. Rifling in the barrel of a canon causes the projectile to spin just as is the case for rifles (hence the name for the grooves in the barrel).  
(credit: Elsie esq., Flickr)

## Problems & Exercises

### Exercise:

#### Problem:

- (a) Calculate the angular momentum of the Earth in its orbit around the Sun.
- (b) Compare this angular momentum with the angular momentum of Earth on its axis.

---

#### Solution:

- (a)  $2.66 \times 10^{40} \text{ kg} \cdot \text{m}^2/\text{s}$

(b)  $7.07 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}$

The angular momentum of the Earth in its orbit around the Sun is  $3.77 \times 10^6$  times larger than the angular momentum of the Earth around its axis.

**Exercise:**

**Problem:**

(a) What is the angular momentum of the Moon in its orbit around Earth?

(b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.

(c) Discuss whether the values found in parts (a) and (b) seem consistent with the fact that tidal effects with Earth have caused the Moon to rotate with one side always facing Earth.

**Exercise:**

**Problem:**

Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What angular momentum is given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

---

**Solution:**

$22.5 \text{ kg} \cdot \text{m}^2/\text{s}$

**Exercise:**

**Problem:**

A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with an angular velocity of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.

**Exercise:****Problem:**

Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new angular velocity in rpm?

---

**Solution:**

25.3 rpm

**Exercise:****Problem:**

(a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is  $0.400 \text{ kg} \cdot \text{m}^2$ . (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?

**Exercise:****Construct Your Own Problem**

**Problem:**

Consider the Earth-Moon system. Construct a problem in which you calculate the total angular momentum of the system including the spins of the Earth and the Moon on their axes and the orbital angular momentum of the Earth-Moon system in its nearly monthly rotation. Calculate what happens to the Moon's orbital radius if the Earth's rotation decreases due to tidal drag. Among the things to be considered are the amount by which the Earth's rotation slows and the fact that the Moon will continue to have one side always facing the Earth.

**Glossary**

angular momentum

the product of moment of inertia and angular velocity

law of conservation of angular momentum

angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

## Useful Information

This appendix is broken into several tables.

- [\[link\]](#), Important Constants
- [\[link\]](#), Submicroscopic Masses
- [\[link\]](#), Solar System Data
- [\[link\]](#), Metric Prefixes for Powers of Ten and Their Symbols
- [\[link\]](#), The Greek Alphabet
- [\[link\]](#), SI units
- [\[link\]](#), Selected British Units
- [\[link\]](#), Other Units
- [\[link\]](#), Useful Formulae

Symbol	Meaning	Best Value	Approximate Value
$c$	Speed of light in vacuum	$2.99792458 \times 10^8 \text{ m/s}$	$3.00 \times 10^8 \text{ m/s}$
$G$	Gravitational constant	$6.67408(31) \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$	$6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$
$N_A$	Avogadro's number	$6.02214129(27) \times 10^{23}$	$6.02 \times 10^{23}$
$k$	Boltzmann's constant	$1.3806488(13) \times 10^{-23} \text{ J/K}$	$1.38 \times 10^{-23} \text{ J/K}$
$R$	Gas constant	$8.3144621(75) \text{ J/mol} \cdot \text{K}$	$8.31 \text{ J/mol} \cdot \text{K} = 1.99 \text{ cal/mol} \cdot \text{K} =$
$\sigma$	Stefan-Boltzmann constant	$5.670373(21) \times 10^{-8} \text{ W/m}^2 \cdot \text{K}$	$5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}$
$k$	Coulomb force constant	$8.987551788... \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$	$8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$
$q_e$	Charge on electron	$-1.602176565(35) \times 10^{-19} \text{ C}$	$-1.60 \times 10^{-19} \text{ C}$
$\epsilon_0$	Permittivity of free space	$8.854187817... \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$	$8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$
$\mu_0$	Permeability of free space	$4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$	$1.26 \times 10^{-6} \text{ T} \cdot \text{m/A}$
$h$	Planck's constant	$6.62606957(29) \times 10^{-34} \text{ J} \cdot \text{s}$	$6.63 \times 10^{-34} \text{ J} \cdot \text{s}$

Important Constants<sup>[footnote]</sup>

Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, [www.physics.nist.gov/cuu](http://www.physics.nist.gov/cuu) (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

Symbol	Meaning	Best Value	Approximate Value
$m_e$	Electron mass	$9.10938291(40) \times 10^{-31}\text{kg}$	$9.11 \times 10^{-31}\text{kg}$
$m_p$	Proton mass	$1.672621777(74) \times 10^{-27}\text{kg}$	$1.6726 \times 10^{-27}\text{kg}$
$m_n$	Neutron mass	$1.674927351(74) \times 10^{-27}\text{kg}$	$1.6749 \times 10^{-27}\text{kg}$
u	Atomic mass unit	$1.660538921(73) \times 10^{-27}\text{kg}$	$1.6605 \times 10^{-27}\text{kg}$

#### Submicroscopic Masses<sup>[footnote]</sup>

Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, [www.physics.nist.gov/cuu](http://www.physics.nist.gov/cuu) (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

<b>Sun</b>	mass	$1.99 \times 10^{30}\text{kg}$
	average radius	$6.96 \times 10^8\text{m}$
	Earth-sun distance (average)	$1.496 \times 10^{11}\text{m}$
<b>Earth</b>	mass	$5.9736 \times 10^{24}\text{kg}$
	average radius	$6.376 \times 10^6\text{m}$
	orbital period	$3.16 \times 10^7\text{s}$





Epsilon	Ε	ε	Lambda	Λ	λ	Rho	Ρ	ρ	Psi	Ψ	ψ
Zeta	Ζ	ζ	Mu	Μ	μ	Sigma	Σ	σ	Omega	Ω	ω

The Greek Alphabet

	Entity	Abbreviation	Name
Fundamental units	Length	m	meter
	Mass	kg	kilogram
	Time	s	second
	Current	A	ampere
Supplementary unit	Angle	rad	radian
Derived units	Force	$N = kg \cdot m/s^2$	newton
	Energy	$J = kg \cdot m^2/s^2$	joule
	Power	$W = J/s$	watt
	Pressure	$Pa = N/m^2$	pascal
	Frequency	$Hz = 1/s$	hertz
	Electronic potential	$V = J/C$	volt
	Capacitance	$F = C/V$	farad
	Charge	$C = s \cdot A$	coulomb
	Resistance	$\Omega = V/A$	ohm

	Entity	Abbreviation	Name
	Magnetic field	$T = N/(A \cdot m)$	tesla
	Nuclear decay rate	$Bq = 1/s$	becquerel

#### SI Units

Length	1 inch (in.) = 2.54 cm (exactly)
	1 foot (ft) = 0.3048 m
	1 mile (mi) = 1.609 km
Force	1 pound (lb) = 4.448 N
Energy	1 British thermal unit (Btu) = $1.055 \times 10^3$ J
Power	1 horsepower (hp) = 746 W
Pressure	$1 \text{ lb/in}^2 = 6.895 \times 10^3$ Pa

#### Selected British Units

Length	1 light year (ly) = $9.46 \times 10^{15}$ m
	1 astronomical unit (au) = $1.50 \times 10^{11}$ m
	1 nautical mile = 1.852 km
	1 angstrom( $\text{\AA}$ ) = $10^{-10}$ m
Area	1 acre (ac) = $4.05 \times 10^3$ m <sup>2</sup>
	1 square foot (ft <sup>2</sup> ) = $9.29 \times 10^{-2}$ m <sup>2</sup>
	1 barn ( <i>b</i> ) = $10^{-28}$ m <sup>2</sup>
Volume	1 liter ( <i>L</i> ) = $10^{-3}$ m <sup>3</sup>

	1 U.S. gallon (gal) = $3.785 \times 10^{-3} \text{ m}^3$
Mass	1 solar mass = $1.99 \times 10^{30} \text{ kg}$
	1 metric ton = $10^3 \text{ kg}$
	1 atomic mass unit ( $u$ ) = $1.6605 \times 10^{-27} \text{ kg}$
Time	1 year ( $y$ ) = $3.16 \times 10^7 \text{ s}$
	1 day ( $d$ ) = 86,400 s
Speed	1 mile per hour (mph) = 1.609 km/h
	1 nautical mile per hour (naut) = 1.852 km/h
Angle	1 degree ( $^\circ$ ) = $1.745 \times 10^{-2} \text{ rad}$
	1 minute of arc ( $'$ ) = 1/60 degree
	1 second of arc ( $''$ ) = 1/60 minute of arc
	1 grad = $1.571 \times 10^{-2} \text{ rad}$
Energy	1 kiloton TNT (kT) = $4.2 \times 10^{12} \text{ J}$
	1 kilowatt hour ( $\text{kW} \cdot \text{h}$ ) = $3.60 \times 10^6 \text{ J}$
	1 food calorie (kcal) = 4186 J
	1 calorie (cal) = 4.186 J
	1 electron volt (eV) = $1.60 \times 10^{-19} \text{ J}$
Pressure	1 atmosphere (atm) = $1.013 \times 10^5 \text{ Pa}$
	1 millimeter of mercury (mm Hg) = 133.3 Pa
	1 torricelli (torr) = 1 mm Hg = 133.3 Pa
Nuclear decay rate	1 curie (Ci) = $3.70 \times 10^{10} \text{ Bq}$

#### Other Units

Circumference of a circle with radius $r$ or diameter $d$	$C = 2\pi r = \pi d$
Area of a circle with radius $r$ or diameter $d$	$A = \pi r^2 = \pi d^2/4$
Area of a sphere with radius $r$	$A = 4\pi r^2$

Volume of a sphere with radius  $r$

$$V = (4/3)(\pi r^3)$$

Useful Formulae

## Glossary of Key Symbols and Notation

In this glossary, key symbols and notation are briefly defined.

Symbol	Definition
any symbol	average (indicated by a bar over a symbol— e.g., $\bar{v}$ is average velocity)
$^{\circ}\text{C}$	Celsius degree
$^{\circ}\text{F}$	Fahrenheit degree
//	parallel
$\perp$	perpendicular
$\propto$	proportional to
$\pm$	plus or minus

Symbol	Definition
0	zero as a subscript denotes an initial value
$\alpha$	alpha rays
$\alpha$	angular acceleration
$\alpha$	temperature coefficient(s) of resistivity
$\beta$	beta rays
$\beta$	sound level
$\beta$	volume coefficient of expansion
$\beta^{-}$	electron emitted in nuclear beta decay
$\beta^{+}$	positron decay
$\gamma$	gamma rays

Symbol	Definition
$\gamma$	surface tension
$\gamma = 1/\sqrt{1 - v^2/c^2}$	a constant used in relativity
$\Delta$	change in whatever quantity follows
$\delta$	uncertainty in whatever quantity follows
$\Delta E$	change in energy between the initial and final orbits of an electron in an atom
$\Delta E$	uncertainty in energy
$\Delta m$	difference in mass between initial and final products
$\Delta N$	number of decays that occur
$\Delta p$	change in momentum

Symbol	Definition
$\Delta p$	uncertainty in momentum
$\Delta PE_g$	change in gravitational potential energy
$\Delta\theta$	rotation angle
$\Delta s$	distance traveled along a circular path
$\Delta t$	uncertainty in time
$\Delta t_0$	proper time as measured by an observer at rest relative to the process
$\Delta V$	potential difference
$\Delta x$	uncertainty in position
$\epsilon_0$	permittivity of free space
$\eta$	viscosity



Symbol	Definition
$\theta$	angle between the force vector and the displacement vector
$\theta$	angle between two lines
$\theta$	contact angle
$\theta$	direction of the resultant
$\theta_b$	Brewster's angle
$\theta_c$	critical angle
$\kappa$	dielectric constant
$\lambda$	decay constant of a nuclide
$\lambda$	wavelength
$\lambda_n$	wavelength in a medium

Symbol	Definition
$\mu_0$	permeability of free space
$\mu_k$	coefficient of kinetic friction
$\mu_s$	coefficient of static friction
$\nu_e$	electron neutrino
$\pi^+$	positive pion
$\pi^-$	negative pion
$\pi^0$	neutral pion
$\rho$	density
$\rho_c$	critical density, the density needed to just halt universal expansion
$\rho_{\text{fl}}$	fluid density

Symbol	Definition
$\rho_{\text{obj}}$	average density of an object
$\rho/\rho_{\text{w}}$	specific gravity
$\tau$	characteristic time constant for a resistance and inductance (RL) or resistance and capacitance (RC) circuit
$\tau$	characteristic time for a resistor and capacitor (RC) circuit
$\tau$	torque
$\Upsilon$	upsilon meson
$\Phi$	magnetic flux
$\phi$	phase angle
$\Omega$	ohm (unit)
$\omega$	angular velocity

Symbol	Definition
A	ampere (current unit)
$A$	area
$A$	cross-sectional area
$A$	total number of nucleons
$a$	acceleration
$a_B$	Bohr radius
$a_c$	centripetal acceleration
$a_t$	tangential acceleration
AC	alternating current
AM	amplitude modulation

Symbol	Definition
atm	atmosphere
$B$	baryon number
$B$	blue quark color
$B$	antiblue (yellow) antiquark color
$b$	quark flavor bottom or beauty
$B$	bulk modulus
$B$	magnetic field strength
$B_{\text{int}}$	electron's intrinsic magnetic field
$B_{\text{orb}}$	orbital magnetic field
BE	binding energy of a nucleus—it is the energy required to completely disassemble it into separate protons and neutrons

Symbol	Definition
$BE/A$	binding energy per nucleon
Bq	becquerel—one decay per second
$C$	capacitance (amount of charge stored per volt)
$C$	coulomb (a fundamental SI unit of charge)
$C_p$	total capacitance in parallel
$C_s$	total capacitance in series
CG	center of gravity
CM	center of mass
$c$	quark flavor charm
$c$	specific heat

Symbol	Definition
$c$	speed of light
Cal	kilocalorie
cal	calorie
$COP_{hp}$	heat pump's coefficient of performance
$COP_{ref}$	coefficient of performance for refrigerators and air conditioners
$\cos \theta$	cosine
$\cot \theta$	cotangent
$\csc \theta$	cosecant
$D$	diffusion constant
$d$	displacement

Symbol	Definition
$d$	quark flavor down
dB	decibel
$d_i$	distance of an image from the center of a lens
$d_o$	distance of an object from the center of a lens
DC	direct current
$E$	electric field strength
$\varepsilon$	emf (voltage) or Hall electromotive force
emf	electromotive force
$E$	energy of a single photon
$E$	nuclear reaction energy



Symbol	Definition
$E$	relativistic total energy
$E$	total energy
$E_0$	ground state energy for hydrogen
$E_0$	rest energy
EC	electron capture
$E_{\text{cap}}$	energy stored in a capacitor
Eff	efficiency—the useful work output divided by the energy input
Eff <sub>C</sub>	Carnot efficiency
$E_{\text{in}}$	energy consumed (food digested in humans)
$E_{\text{ind}}$	energy stored in an inductor

Symbol	Definition
$E_{\text{out}}$	energy output
$e$	emissivity of an object
$e^+$	antielectron or positron
eV	electron volt
F	farad (unit of capacitance, a coulomb per volt)
F	focal point of a lens
<b>F</b>	force
$F$	magnitude of a force
$F$	restoring force
$F_{\text{B}}$	buoyant force

Symbol	Definition
$F_c$	centripetal force
$F_i$	force input
$\mathbf{F}_{\text{net}}$	net force
$F_o$	force output
FM	frequency modulation
$f$	focal length
$f$	frequency
$f_0$	resonant frequency of a resistance, inductance, and capacitance (RLC) series circuit
$f_0$	threshold frequency for a particular material (photoelectric effect)

Symbol	Definition
$f_1$	fundamental
$f_2$	first overtone
$f_3$	second overtone
$f_B$	beat frequency
$f_k$	magnitude of kinetic friction
$f_s$	magnitude of static friction
$G$	gravitational constant
$G$	green quark color
$\bar{G}$	antigreen (magenta) antiquark color

Symbol	Definition
$g$	acceleration due to gravity
$g$	gluons (carrier particles for strong nuclear force)
$h$	change in vertical position
$h$	height above some reference point
$h$	maximum height of a projectile
$h$	Planck's constant
$hf$	photon energy
$h_i$	height of the image
$h_o$	height of the object
$I$	electric current

Symbol	Definition
$I$	intensity
$I$	intensity of a transmitted wave
$I$	moment of inertia (also called rotational inertia)
$I_0$	intensity of a polarized wave before passing through a filter
$I_{\text{ave}}$	average intensity for a continuous sinusoidal electromagnetic wave
$I_{\text{rms}}$	average current
J	joule
$J/\Psi$	Joules/psi meson
K	kelvin
$k$	Boltzmann constant

Symbol	Definition
$k$	force constant of a spring
$K_{\alpha}$	x rays created when an electron falls into an $n = 1$ shell vacancy from the $n = 3$ shell
$K_{\beta}$	x rays created when an electron falls into an $n = 2$ shell vacancy from the $n = 3$ shell
kcal	kilocalorie
KE	translational kinetic energy
KE + PE	mechanical energy
$\text{KE}_e$	kinetic energy of an ejected electron
$\text{KE}_{\text{rel}}$	relativistic kinetic energy
$\text{KE}_{\text{rot}}$	rotational kinetic energy
KE	thermal energy

Symbol	Definition
kg	kilogram (a fundamental SI unit of mass)
$L$	angular momentum
L	liter
$L$	magnitude of angular momentum
$L$	self-inductance
$\ell$	angular momentum quantum number
$L_{\alpha}$	x rays created when an electron falls into an $n = 2$ shell from the $n = 3$ shell
$L_e$	electron total family number
$L_{\mu}$	muon family total number
$L_{\tau}$	tau family total number



Symbol	Definition
$L_f$	heat of fusion
$L_f$ and $L_v$	latent heat coefficients
$L_{orb}$	orbital angular momentum
$L_s$	heat of sublimation
$L_v$	heat of vaporization
$L_z$	z - component of the angular momentum
$M$	angular magnification
$M$	mutual inductance
m	indicates metastable state
$m$	magnification

Symbol	Definition
$m$	mass
$m$	mass of an object as measured by a person at rest relative to the object
m	meter (a fundamental SI unit of length)
$m$	order of interference
$m$	overall magnification (product of the individual magnifications)
$m(^A\text{X})$	atomic mass of a nuclide
MA	mechanical advantage
$m_e$	magnification of the eyepiece
$m_e$	mass of the electron
$m_\ell$	angular momentum projection quantum number

Symbol	Definition
$m_n$	mass of a neutron
$m_o$	magnification of the objective lens
mol	mole
$m_p$	mass of a proton
$m_s$	spin projection quantum number
$N$	magnitude of the normal force
N	newton
<b>N</b>	normal force
$N$	number of neutrons
$n$	index of refraction

Symbol	Definition
$n$	number of free charges per unit volume
$N_A$	Avogadro's number
$N_r$	Reynolds number
$N \cdot m$	newton-meter (work-energy unit)
$N \cdot m$	newtons times meters (SI unit of torque)
OE	other energy
$P$	power
$P$	power of a lens
$P$	pressure
<b>p</b>	momentum

Symbol	Definition
$p$	momentum magnitude
$p$	relativistic momentum
$\mathbf{p}_{\text{tot}}$	total momentum
$\mathbf{p}'_{\text{tot}}$	total momentum some time later
$P_{\text{abs}}$	absolute pressure
$P_{\text{atm}}$	atmospheric pressure
$P_{\text{atm}}$	standard atmospheric pressure
PE	potential energy
PE <sub>el</sub>	elastic potential energy
PE <sub>elec</sub>	electric potential energy

Symbol	Definition
$PE_s$	potential energy of a spring
$P_g$	gauge pressure
$P_{in}$	power consumption or input
$P_{out}$	useful power output going into useful work or a desired, form of energy
$Q$	latent heat
$Q$	net heat transferred into a system
$Q$	flow rate—volume per unit time flowing past a point
$+Q$	positive charge
$-Q$	negative charge

Symbol	Definition
$q$	electron charge
$q_p$	charge of a proton
$q$	test charge
QF	quality factor
$R$	activity, the rate of decay
$R$	radius of curvature of a spherical mirror
$R$	red quark color
$R$	antired (cyan) quark color
$R$	resistance
R	resultant or total displacement

Symbol	Definition
$R$	Rydberg constant
$R$	universal gas constant
$r$	distance from pivot point to the point where a force is applied
$r$	internal resistance
$r_{\perp}$	perpendicular lever arm
$r$	radius of a nucleus
$r$	radius of curvature
$r$	resistivity
r or rad	radiation dose unit
rem	roentgen equivalent man



Symbol	Definition
rad	radian
RBE	relative biological effectiveness
RC	resistor and capacitor circuit
rms	root mean square
$r_n$	radius of the $n$ th H-atom orbit
$R_p$	total resistance of a parallel connection
$R_s$	total resistance of a series connection
$R_s$	Schwarzschild radius
$S$	entropy
<b>S</b>	intrinsic spin (intrinsic angular momentum)

Symbol	Definition
$S$	magnitude of the intrinsic (internal) spin angular momentum
$S$	shear modulus
$S$	strangeness quantum number
$s$	quark flavor strange
s	second (fundamental SI unit of time)
$s$	spin quantum number
<b>s</b>	total displacement
$\sec \theta$	secant
$\sin \theta$	sine
$s_z$	z-component of spin angular momentum

Symbol	Definition
$T$	period—time to complete one oscillation
$T$	temperature
$T_c$	critical temperature—temperature below which a material becomes a superconductor
$T$	tension
T	tesla (magnetic field strength $B$ )
$t$	quark flavor top or truth
$t$	time
$t_{1/2}$	half-life—the time in which half of the original nuclei decay
$\tan \theta$	tangent
$U$	internal energy

Symbol	Definition
$u$	quark flavor up
u	unified atomic mass unit
<b>u</b>	velocity of an object relative to an observer
<b>u'</b>	velocity relative to another observer
$V$	electric potential
$V$	terminal voltage
V	volt (unit)
$V$	volume
<b>v</b>	relative velocity between two observers
$v$	speed of light in a material

Symbol	Definition
$\mathbf{v}$	velocity
$\mathbf{v}$	average fluid velocity
$V_B - V_A$	change in potential
$\mathbf{v}_d$	drift velocity
$V_p$	transformer input voltage
$V_{\text{rms}}$	rms voltage
$V_s$	transformer output voltage
$\mathbf{v}_{\text{tot}}$	total velocity
$v_w$	propagation speed of sound or other wave
$\mathbf{v}_w$	wave velocity

Symbol	Definition
$W$	work
$W$	net work done by a system
$W$	watt
$w$	weight
$w_{\text{fl}}$	weight of the fluid displaced by an object
$W_{\text{c}}$	total work done by all conservative forces
$W_{\text{nc}}$	total work done by all nonconservative forces
$W_{\text{out}}$	useful work output
$X$	amplitude
$X$	symbol for an element

Symbol	Definition
${}^Z_A X_N$	notation for a particular nuclide
$x$	deformation or displacement from equilibrium
$x$	displacement of a spring from its undeformed position
$x$	horizontal axis
$X_C$	capacitive reactance
$X_L$	inductive reactance
$x_{\text{rms}}$	root mean square diffusion distance
$y$	vertical axis
$Y$	elastic modulus or Young's modulus
$Z$	atomic number (number of protons in a nucleus)

Symbol	Definition
$Z$	impedance